



Meta Didactic-Mathematical Knowledge of Teachers: Criteria for The Reflection and Assessment on Teaching Practice

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ABSTRACT

The objective of this study is to demonstrate that the criteria of didactical suitability, proposed by the theoretical framework known as the Onto-Semiotic Approach (OSA) of mathematical knowledge and instruction, are powerful tools for organizing the reflection and assessment of instruction processes carried out by mathematics teachers. To this aim, the results of a multiple case study are presented which prove that when teachers are faced with the task of evaluating processes of instruction, they employ-either implicitly or explicitly- the OSA criteria of didactical suitability (epistemic, cognitive, affective, mediational, interactional and ecological). Therefore, the explicit application of said criteria in teachers' education cycles contributes to the development of the didactic analysis competence that is necessary for teachers to be able to reflect on their own practices.

Keywords: teacher training, didactic-mathematical knowledge, competences, didactic analysis, didactical suitability

INTRODUCTION

The study of didactic and mathematical knowledge and competences that a teacher should have in order to manage accordingly how their students learn is a topic that has been extensively investigated, thus generating various proposed models with which to characterize said knowledge and proficiencies of the teacher (e.g., Shulman, 1986; 1987; Rowland, Huckstep & Thwaites, 2005; Hill, Ball & Schilling, 2008; Schoenfeld & Kilpatrick, 2008). Of the elements put forward in such models, this article takes particular interest in the teacher's reflection on their practice.

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State of the literature

- Internationally, there are several models oriented to characterize and develop the didactic-mathematical knowledge and competencies of mathematics teachers.
- In general, these models presented in the international research community on mathematics education, pose new categories of teacher knowledge without elaborating guidelines or criteria that are needed to develop them.
- The theoretical contribution called Onto-Semiotic Approach (OSA) proposed the notion of didactical suitability, which allows to qualify an instruction process as suitable.

Contribution of this paper to the literature

- The Didactic-Mathematical Knowledge and Competences model (DMKC), aims to develop knowledge and competencies that enable the design, the implementation and the evaluation of suitable processes of teaching and learning of mathematics.
- The Didactic-Mathematical Knowledge and Competences model (DMKC), besides serving for the organization of the categories of teachers' knowledge, it provides criteria and specific guidelines that allow its operationalization.
- It is considered that the use of the didactical suitability criteria in the teacher education processes could be a powerful methodological tool to promote and support teachers' reflection on their own practice.

The teacher's reflection on their own practice is a key aspect in the teacher education, in fact as Schön (1983) highlights, the teacher is a reflective, rational subject who makes decisions, formulates opinions, has beliefs and generates their own routine for their professional development. In the same way, the teacher's thoughts significantly influence their own behavior, even defining it (Schön, 1987). According to Schoenfeld and Kilpatrick (2008, p. 348), "Once it is habitual, reflection can become the principal mechanism for improving one's teaching practice".

Similarly, Pino-Fan, Assis and Castro (2015) put forward a model for the teacher's didactic-mathematical knowledge, which enables the characterization of the teacher's knowledge on the basis of three dimensions: mathematical, didactical and meta didactic-mathematical. For this investigation, we focused on the third dimension, which refers to:

"the knowledge needed by teachers to: reflect on their own practice, identify and analyze the set of norms and meta-norms that regulate the teaching and learning processes of mathematics, and assess the didactic suitability in order to find potential improvements in the both design and implementation stages of such processes of study". (Pino-Fan, Godino & Font, 2016, p. 8)

In order to develop this meta didactic-mathematical dimension, the theoretical model known as the Onto-Semiotic Approach (OSA) of knowledge and mathematical instruction (Godino, Batanero & Font, 2007) has proposed different theoretical constructs, in particular for the assessment of mathematics' teaching and learning processes. Thus, the notion of

didactic suitability is put forward as an essential tool; it is understood as the degree to which said process (or a part of) combines certain characteristics in order to be classified suitable (optimal or appropriate) for the adaptation between the personal meanings obtained by students (learning), and the intended or implemented institutional meanings (teaching), taking into consideration the circumstances and the available resources (environment). The didactical suitability notion is broken down into six specific aptitudes (epistemic, cognitive, affective, interactional, mediational and ecological suitability). For each of these six suitability types, a system of components and characteristics associated is proposed.

The aim of this investigation is to show that the criteria of didactical suitability are powerful tools for organizing the reflection and assessment of both one's own and another's instruction processes, including in training contexts where the teachers are not provided with explicit guidelines for such evaluation.

The structure of this article is as follows: subsequent to this introduction, the second section will explain the model known as Didactic-Mathematical Knowledge and Competences (DMKC model), which is based on the theoretical constructs of the OSA. In particular, the article will explain the key competence of this model known as the competence of 'didactic analysis and intervention', which is made up of four sub-competences, one of them being the assessment of the didactical suitability of instruction processes. In the third section, the model of didactical analysis of instruction processes developed by the OSA and its relation to the DMKC model will be explained. The development of the competence of didactic analysis and intervention enables teachers to carry out the types or levels of didactical analysis of the model proposed by the OSA and, at the same time, the training mechanisms devised for the teaching and learning of these types of didactic analysis contribute to the development of said competence and to the acquisition of the teacher's knowledge as outlined by the DMKC model. In the fourth section, some of the empirical studies related to the DMKC model will be presented. The fifth section will explain a series of research projects, a Naturalist Inquiry and training courses that, on one hand, have enabled the development of the DMKC model and, on the other hand, have allowed this same model to be trialed in empirical observations. The empirical evidence revealed different patterns, one of which being that when opinions are clearly evaluative they are -implicitly or explicitly- organized by means of some characteristics of the didactical suitability criteria components. It was also observed that the positive evaluation of these characteristics is based on the implicit or explicit assumption that there are certain tendencies within mathematics teaching which demonstrate how quality mathematics teaching should be. The sixth, seventh and eight sections will describe an investigation of a Naturalist Inquiry multiple case study which was designed to confirm the two aforementioned patterns.

THEORETICAL FRAMEWORK

Investigations on the competences, mathematical knowledge and professional development of the teacher has gained international relevance over recent years and has

brought to light its complexity and the limitations of knowledge produced by said investigations (Sullivan & Wood, 2008; Silverman & Thompson, 2008; Stahnke, Schueler & Roesken-Winter, 2016). Thus, there is a growing need in the field of research, for relating theoretical models of teacher's education with teaching practice, a need that is also present in the training programs and teaching innovation projects.

The need for theoretical tools for the analysis of teacher's practice is derived, for example, of the fact that it is not enough to contemplate, in the training of teachers, opportunities for these reflect on their practice, because they need theoretical tools which allow to draw your attention to important aspects of the teaching and learning processes of mathematics. On the other hand, research has provided examples to demonstrate that these theoretical tools can be taught to teachers and future teachers (e.g., Turner, 2012; Sun & van Es, 2015; Giménez, Font & Vanegas, 2013; Seckel, 2016).

From different perspectives of investigation, each directly related to teaching practices, emphasis is placed on the way in which teachers' knowledge of the mathematical content is made evident by the employment of good practices in their classes. Amongst those, we can highlight: 1) the works of Rowland *et al* (Rowland, Huckstep & Thwaites, 2005; Liston, 2015) and their proposal of four knowledge categories (foundation, transformation, connection and contingency) that characterize the knowledge of the actively instructing teacher; 2) the contributions derived from the Lesson Study methodology (Fernández & Yoshida, 2004) which include a collaborative analysis model employed by the teachers in order to plan, implement, observe and reflect on their mathematics classes; 3) Davis' contributions and collaborators (Davis, 2008; Davis & Renert, 2013) in the framework of his proposed Concept Study, which combines elements of the two relevant approaches for the investigation of mathematics education: "concept analysis" and "lesson study". Each of these proposals commonly focus on the specific nature granted to mathematical knowledge for teaching. This focus gives them the edge over other investigations of a more general character, which differentiate general didactic knowledge from mathematical knowledge as a scientific discipline.

The previously mentioned investigations have a shared objective of improving mathematical practices in the classroom, with a focus on the complexity *a priori* of the mathematical objects, or rather on the teacher's mathematical knowledge which is at play in the management of the complex mathematical objects. This is a shared purpose by other approaches oriented to the professional knowledge, the aim of which is to recognize the actions that allow the mathematics teacher to successfully develop their profession. Mason (2002), for example, highlights the importance of the competence referred to as "looking with sense" the mathematical thinking of the students (Fernández, Llinares & Valls, 2012; Mason, 2002). Such competence, allows the mathematics teacher to see contexts of teaching and learning mathematics in a professional way that can be differentiated from the way in which a non-mathematics teacher would look at the situation. In the same way, other authors emphasize the importance of the didactic analysis competence (Gómez, 2006), which allow

the teacher, primarily, to identify and organize the multiple meanings of the concept they wish to teach and, secondly, to select those meanings to be studied in the instruction processes.

In order to be able to systematically carry out a didactic analysis that allows the description, explanation and evaluation of the instruction processes, it is necessary to rely on tools specially designed to deal with the complexity of mathematics and the complexity of the instruction processes. In order to address this necessity, the investigation of mathematics education has given rise to specific theoretical frameworks that provide analysis tools. The Onto-Semiotic Approach (OSA) of mathematics instruction is one of these.

Didactic-Mathematical Knowledge and Competences Model (DMKC)

Within the Onto-Semiotic Approach (OSA) of mathematical knowledge and instruction (Godino, Batanero & Font, 2007), a theoretical model of the mathematics teacher's knowledge (Pino-Fan, Assis & Castro, 2015; Pino-Fan, Godino & Font, 2016), known as the Didactic-Mathematical Knowledge model (DMK model), has been developed. As stated by these authors, one of the aspects of the development of said model is the interconnectedness of the notion of the teacher's 'knowledge' with that of their 'competence'. Further, also with by the OSA framework, significant research has been carried out on the competences of the mathematics teacher (Font, 2011; Rubio, 2012; Giménez, Font & Vanegas, 2013; Seckel, 2016; Pochulu, Font & Rodríguez, 2016), which has also brought to light the need for such a model on teachers' knowledge in order to evaluate and develop their competences. Both of these research topics have converged to create the model known as Didactic-Mathematical Knowledge and Competences of the mathematics teacher (DMKC model) (Godino, Giacomone, Batanero & Font, 2016; *in press*).

Within the DMKC model it is considered that the two key competences of the mathematics teacher are: 1) the mathematical competency; and 2) the didactical analysis and intervention competency, whose fundamental nucleus (Font, 2011; Pino-Fan, Assis & Castro, 2015) consists of designing, applying and assessing the sequences of one's own learning, and that of others, by means of didactic analysis techniques and criteria of quality, in order to establish cycles of planning, implementation, evaluation and to put forward proposals for improvement. In order to develop such competence, the teacher requires, on one hand, knowledge that allows them to describe and explain what has happened in the process of teaching and learning –the didactical dimension of the DMK model (Pino-Fan, Assis & Castro, 2015), as a component of the DMKC model– and, on the other hand, they require knowledge to assess what has taken place and to make suggestions for the improvement of future implementations –the meta didactic-mathematical dimension of the DMK model (Pino-Fan, Assis & Castro, 2015)–. This general competence is formed of different sub-competences: 1) analysis of mathematical activity; 2) analysis and management of interactions and conflicts; 3) analysis of norms and metanorms and 4) assessment of the

didactical suitability of instruction processes. In this article, we will focus on the latter sub-competence.

Sub-competence for the assessment of the didactical suitability of instruction processes

The characterization of the aforementioned didactical analysis and intervention competence requires tools of description and explanation such as those detailed in Rubio's research (2012) on the analysis of mathematical activity, as well as tools for assessment, such as those presented in Ramos and Font's research (2008). In this latest investigation it is demonstrated that, even when teachers are not familiar with the criteria of didactical suitability, its components and characteristics, if they are put in a situation in which they must evaluate a proposal of didactical innovation that could involve them as part of a group, the teachers implicitly use the criteria to make a positive or negative assessment.

To facilitate the assessment of instruction processes, the OSA recommends the notion of didactical suitability as an essential tool. Once a specific topic is established in a concrete educational context, the notion of didactical suitability (Breda, Font & Lima, 2015a; Sullivan, Knott & Yang, 2015) enables us to answer questions such as: What grade of didactical suitability of the teaching and learning process is applied? What changes should be introduced to the design and implementation of the instruction process in order to increase didactical suitability in future implementations?

The didactical suitability of an instruction process is thus defined as the extent to which such process (or a part of) brings together certain characteristics that justify it being qualified as *suitable* (optimal or appropriate) for the adaptation amongst the personal meanings achieved by the students (learning), and the intended or implemented institutional meanings (teaching) taking the available resources (environment) into consideration. The notion of didactical suitability can be subdivided into six specific categories:

1. Epistemic Suitability refers to the teaching of 'good mathematics'. In order to achieve this, in addition to considering the approved curriculum, the intention is to refer to institutional mathematics that have been incorporated into the curriculum.
2. Cognitive Suitability refers to the extent to which applied/desired learning is within the parameters of the students' potential development, as well as the correlation between what the students indeed learn and the applied/desired learning.
3. Interactional Suitability is the extent to which the means of interaction allow conflicts of meaning to be identified and resolved and how interaction methods favor autonomous learning.
4. Mediational Suitability is the degree of availability and aptness of time and material resources necessary for the development of teaching-learning processes.

5. Affective Suitability refers to the degree of the students' involvement (interest, motivation) in the study processes.
6. Ecological Suitability is the extent to which the process of study is adapted to the center's educational project, the curricular norms and the social environment etc.

For each of these six categories, or criteria, of the didactical suitability, a system of associated components and characteristics is proposed. That is, a system of requirements that allows the complete, well-balanced evaluation (or self-evaluation) of the elements that, a priori, lead to a quality instruction process in the field of mathematics.

A model of analysis for instructional processes

The OSA considers five levels of analysis for instructional processes (Font, Planas & Godino, 2010; Pochulu & Font, 2011; Contreras, García & Font, 2012; Pino-Fan, Assis & Godino, 2015): 1) The identification of mathematical practices; 2) The elaboration of the configurations of mathematical objects and processes; 3) The analysis of didactical trajectories and interactions; 4) The identification of a system of norms and metanorms; and 5) The assessment of the didactical suitability of the instruction process.

The first level of analysis explores the mathematical practices executed in a mathematical instruction process. The second level of analysis focuses on both the mathematical objects and processes that feature in the execution of the practices, as well as on the objects and processes that arise from the practices. The third level of didactical analysis is, above all, directed towards the description of interaction patterns, to didactical configurations and to their sequential expression in didactical trajectories. The configurations and trajectories are conditioned and reinforced by a scheme of norms and metanorms; the fourth level of analysis studies this same scheme.

The first four levels of analysis are tools for a descriptive-explicative didactic, while the fifth focuses on the assessment of didactical suitability. This last level is based on the previous four analysis levels, and it is a recapitulation orientated towards the identification of potential improvements for the instruction process in new implementations.

The didactic analysis model put forward by the OSA incorporates aspects of both the so-called epistemic approach and the sociocultural theories. On one hand, the analysis of the practices, and mathematical objects and processes makes it possible to describe the mathematics of the process of the instruction analyzed. At the same time, the analysis of both the interactions and the normative dimension enables the interaction produced during the instruction process and its regulating norms to be described. Finally, the suitability criteria include the incorporation of an axiological rationale within mathematics education which enables the analysis, criticism, justification of the choice of means and the aims, the justification of change, etc.

The development of the didactical analysis and intervention competence allows teachers to carry out the levels of didactic analysis put forward by the OSA and, at the same

time, the training mechanisms designed for the teaching and learning of these didactic analysis levels contribute to the development of such competence and to the acquisition of the teacher's knowledge explored in the DMKC model (Rubio, 2012; Seckel, 2016; Pino-Fan, Godino & Font, 2016).

Empirical studies related to the DMKC model

Employing the constructs of the OSA as a theoretical framework, a series of research projects including a Naturalist Inquiry and training courses has been carried out enabling, on one hand, the development of the DMKC model and, on the other hand, making it possible to trial this model in empirical situations.

In different investigations and training contexts, training cycles have been designed and implemented in order for teachers (or future teachers) to develop the competences and to acquire the knowledge outlined by the model (e.g., Alsina & Domingo, 2010; Rubio, 2012; Pochulu, Font & Rodríguez, 2016; Seckel, 2016). That is, cycles of training which are intended to teach the participants some (or all) of the types (or levels) of didactic analysis outlined in the didactic analysis model and put forward by the OSA (Font, Planas & Godino, 2010; Pino-Fan, Assis & Godino, 2015), on the basis that carrying out these didactic analysis techniques enables the development of the key competence of this model: the didactical analysis and intervention competence. It also facilitates the learning of the different types of knowledge explored in the DMKC model. For this reason, the authors of this work have conducted training cycles on numerous occasions (many of them in a workshop format) with the objective of teaching the didactic analysis model put forward by the OSA. In other words, training cycles or workshops designed to be effective learning environments so that: 1) those who attend actively participate, starting with the analysis of classroom situations; and 2) the analysis types proposed by said analysis model are discussed in the group brainstorming session.

In the OSA, such training cycles (workshops) are considered to be *experiments of the development of the teacher's competences and knowledge* (EDTCK) and are a type of Teacher Development Experiment (TDE). According to Simon (2000), the TDEs study the professional development of the prospective teacher or in service teacher, and they are based on the principles of the teaching experiments (Steffe & Thompson, 2000; Cobb, Confrey, di Sessa, Lehrer & Schauble, 2003), meaning that a team of researchers study the teacher's development whilst also promoting their development as part of a continuous cycle of analysis and intervention. This type of investigation also includes the case study.

In the different workshops (EDTCK) for teachers or future teachers of mathematics which were put in place with the aforementioned objectives –two of which are described in Rubio (2012) and Seckel (2016)– we observed, in the phase prior to the explanation and institutionalization of the didactical suitability criteria, the following patterns:

1. When required to give their opinion on a classroom situation instigated by another teacher, the teachers or futures teachers (without having been given any guidelines),

made comments in which aspects of description and/or explanation and/or assessment could be detected.

2. The opinions of these teachers can be considered as evidence of knowledge corresponding to some of the six facets (epistemic, cognitive, affective, interactional, mediational and ecological) from the didactic-mathematical knowledge model (DMK) of the mathematics teacher (which is a component of the DMKC).
3. When the opinions were clearly evaluative, they were organized implicitly or explicitly using some characteristics of the components of the didactical suitability criteria (another component of the DMKC model) proposed by the OSA.
4. The positive assessment of these characteristics is based on the implicit or explicit assumption that there are particular trends in mathematics teaching that indicate how quality mathematics teaching should be. These trends (Guzmán, 2007) are related to the DMKC model given that some of them are the basis from which to propose some of the criteria of didactical suitability.

For this reason, we have considered taking these four observed patterns, related to the DMKC model, and designing empirical studies in order to validate them (in the form of EDTCK experiments and naturalist style case studies). In this article, we describe a Naturalist Inquiry- style multiple case study. The objective of this case study was to corroborate the third and fourth patterns in greater depth.

METHODOLOGY

The aim of this article is to show that the criteria of didactic suitability are powerful tools for organizing the reflection and assessment of both one's own and another's instruction processes, including in training contexts where the teachers are not provided with explicit guidelines for such evaluation. Previously, it is explained that, using the constructs of the OSA, investigations have been developed using different methodological designs (from the Naturalist Inquiry-style studies, to experimental designs), which have made possible the identification of four patterns that support the objective of the article. In this work, we present, in significant detail, one of the aforementioned investigations: A Naturalist Inquiry-style multiple case study formed of 29 teachers in which the latter two of the four patterns are observed. First, we present the context in which the study is developed and, thereafter, the didactic suitability criteria along with its components and characteristics. Following this, we describe the application of said criteria as a means of organizing the teachers' reflections in order to justify the improvement representing its didactical proposal (which is exemplified in the analysis of professor Lopes' reflection). Finally, the results of this case study are presented.

A Naturalist Inquiry – Style Multiple Case Study

With the aim of contributing to the improvement of mathematics teaching, the *Mestrado Profissional em Matemática em Rede Nacional* (PROFMAT) [Professional Master's

Degree in Mathematics of the National Network] was initiated in Brazil in 2011. This Master's degree is aimed at mathematics teachers working in elementary and high school education, especially in state schools. Its objective is to improve their professional training, with emphasis on the mastery and deepening of the mathematical content relevant to their educative service. In this sense, its main objectives (Capes, 2013; SBM, 2013) are:

1. To encourage the improvement of mathematics teaching at all levels.
2. To qualify teachers teaching basic mathematics at postgraduate level, with emphasis on the mastery of mathematical content, offering a professional training course which covers the necessities that arise from the habitual work carried out in the school day.
3. To generate a critical approach to mathematics classes at all levels of primary and secondary education, highlighting that the knowledge of mathematics takes a central role in taking on the demands of modern society.
4. To seek professional development of the educators through the improvement of their training.

Teachers who study for this Master's Degree are expected to put their knowledge into practice for the Final Master's Thesis (FMT) with the objective of discovering the relationship between theoretical and practical knowledge. For this reason, the guidelines of the (PROFMAT) state that the FMT should approach mathematics teaching and elementary education curriculum innovatively and, ideally, the project should be directly applied to the classroom, which contributes to an enriched teaching of discipline (Capes, 2013). Breda and Lima (2016), Breda, Pino-Fan and Font (2016) and Breda, Font and Lima (2016) conducted a study on the proposals of innovation presented in the Final Master's Thesis (FMTs of the PROFMAT, presented in the state Rio Grande del Sur and published in 2013 and 2014, in 29 reports of the Final Master's Thesis in total).

This is based on the assumption that the FMT was a clearly evaluative space given that the teachers are expected to present a proposal for improvement and to carry out a didactical analysis to justify the suitability of such proposal. In this context, the aim was corroborating that the regularities three and four, are also observed.

The qualitative analysis of the data was developed in two stages. In the first, the twenty-nine FMTs were classified according to the type of innovation they proposed and also the phase of the instruction process they considered. The second stage involved the analysis of the reasons provided by the authors of the FMTs to justify the quality of the innovation they proposed. In order to do so, the teachers selected evidence showing the explicit or implicit use of some of the components and characteristics of the different criteria of didactical suitability put forward by the OSA.

The results of the first stage show that the teachers essentially consider three types of innovation: i) mathematical innovation, which considers the incorporation of higher level content in Elementary and High School Education, or rather, the establishment of intra-

mathematical or extra-mathematical connections; ii) innovation of resources –refers to the incorporation of visual materials and manipulatives as well as the incorporation of technology; iii) innovation in values –whereby critical thought and citizenship are introduced; that is, they implicitly assume that the proposals of didactic units which follow certain trends (such as the incorporation of technological resources) are proposals that represent an improvement as far as the habitual way of teaching the content of these units is concerned. In relation to the phase of the instruction process, thirteen of the FMTs presented a didactic sequence plan, eleven carried out the implementation, only one included a change of design, and four did not present any instruction process. More details about the trends upon which the FMT authors base the quality of their proposals can be found in Breda, Font and Lima (2015b; 2016), and Breda, Lima and Pereira (2015). As previously explained, the didactical suitability criteria put forward by the OSA were employed as a theoretical reference for the analysis of the categories used in the FMTs to defend the improvement of mathematics teaching. The effectiveness of the suitability criteria relies on defining a set of observable characteristics that enable the assessment of the grade of suitability of each of the facets of the instruction process. For instance, we all agree that it is necessary to implement “good” mathematics, but what we consider to be “good” can vary significantly. For some criteria, the characteristics are agreed upon relatively easily (for the criteria of suitability of means) whilst for others, as is the case of epistemic suitability, it is more complicated. Godino, Bencomo, Font and Wilhelmi (2007) have contributed a system of characteristics that act as an analysis and evaluation guide for didactical suitability. On the other hand, in Font (2015) the same guide has been used and adapted for the education of mathematics teachers. In this investigation, Font’s (2015) reformulation was used. Detailed below, in the **Tables 1-6** (Font, 2015), are the components of the suitability criteria and, in further detail, the characteristics that put them into effect. With this guide, in the period 2009-2016, more than 400 Final Master’s Thesis (FMT) have been executed in Spain and Ecuador.

Table 1. Components and characteristics of epistemic suitability

Components	Characteristics
Errors	✓ Practices considered mathematically incorrect are not observed.
Ambiguities	✓ Ambiguities that could confuse students are not observed; definitions and procedures are clear and correctly expressed, and adapted to the target level of education; explanations, evidence and demonstrations are suitable for the target level of education, the use of metaphors is controlled, etc.
Diversity of processes	✓ Relevant processes in mathematical activity (modelling, argumentation, problem-solving, connections, etc.) are considered in the sequence of tasks.
Representation	<ul style="list-style-type: none"> ✓ The partial meanings (constituted of definitions, properties, procedures, etc.), are representative samples of the complexity of the mathematical notion chosen to be taught as part of the curriculum. ✓ For one or more partial meanings, a representative sample of problems is provided. ✓ The use of different modes of expression (verbal, graphic, symbolic...), treatments and conversations amongst students are part of one or more of the constituents of partial sense.

Table 2. Components and characteristics of cognitive suitability

Components	Characteristics
Previous knowledge (similar components to those of epistemic suitability)	<ul style="list-style-type: none"> ✓ Students have the necessary previous knowledge to study the topic (that is, they have previously studied or the teacher makes a study plan). ✓ The intended meanings (reasonable difficulty) can be taught through its diverse components.
Adaptation of the curriculum to the individuals' different needs	<ul style="list-style-type: none"> ✓ Development and support activities are included.
Learning	<ul style="list-style-type: none"> ✓ The diverse methods of evaluation demonstrate the application of intended or implemented knowledge/competences.
High cognitive demand	<ul style="list-style-type: none"> ✓ Relevant cognitive processes are activated (generalization, intra-mathematical connections, changes of representations, speculations, etc.) ✓ Metacognitive processes are promoted.

Table 3. Components and characteristics of interactional suitability

Components	Characteristics
Teacher-student interaction	<ul style="list-style-type: none"> ✓ The teacher appropriately presents the topic (clear and well-organized presentation, not speaking too fast, emphasis on the key concept of the topic, etc.) ✓ Students' conflicts of sense are recognized and resolved (students' silence, facial expressions, questions are correctly interpreted and an appropriate survey is conducted, etc.) ✓ The aim is to reach a consensus on the best argument. ✓ Varieties of rhetorical and rational devices are used to involve the students and capture their attention. ✓ The inclusion of students into the class dynamic is facilitated –exclusion is not.
Interaction amongst learners	<ul style="list-style-type: none"> ✓ Dialogue and communication between students is encouraged. ✓ Inclusion in the group is preferred and exclusion is discouraged.
Autonomy	<ul style="list-style-type: none"> ✓ Moments in which students take on responsibility for their study (exploration, formulation and validation) are observed.
Formative evaluation	<ul style="list-style-type: none"> ✓ Systematic observation of the cognitive progress of the students.

Table 4. Components and characteristics of mediational suitability

Components	Characteristics
Material resources (manipulatives, calculators, computers)	<ul style="list-style-type: none"> ✓ The use of manipulatives and technology, which give way to favorable conditions, language, procedures, and arguments, adapted to the intended sense. ✓ Definitions and properties are contextualized and motivated using concrete situations, models, and visualizations.
Number of students, scheduling, classroom conditions	<ul style="list-style-type: none"> ✓ The number and distribution of students enables the desired teaching to take place. ✓ The timetable of the course is appropriate (for example, not all the classes are held late) ✓ The classroom and the distribution of the students is appropriate for the development of the intended instructional method.
Time (for group teaching/tutorials; time for learning)	<ul style="list-style-type: none"> ✓ Accommodating the intended/implemented content to the available time (contact or non-contact hours) ✓ Devotion of time to the most important or central aspects of the topic. ✓ Devotion of time to topic areas that present more difficulty.

Table 5. Components and characteristics of affective suitability

Components	Characteristics
Interests and needs	<ul style="list-style-type: none"> ✓ The selection of tasks that are of interest to the students. ✓ Introduction of scenarios that enable students to evaluate the practicality of mathematics in everyday situations and professional life.
Attitudes	<ul style="list-style-type: none"> ✓ Promoting involvement in activities, perseverance, responsibility, etc. ✓ Reasoning should be done so in a context of equality; the argument will be valued in its own right and not by the person who puts it forward.
Emotions	<ul style="list-style-type: none"> ✓ Promotion of self-esteem, avoiding rejection, phobia or fear of mathematics. ✓ Aesthetic qualities and the precision of mathematics are emphasized.

Table 6. Components and characteristics of ecological suitability

Components	Characteristics
Adaptation to the curriculum	<ul style="list-style-type: none"> ✓ The content, its implementation and evaluation, correspond to the curricular plan.
Intra/Interdisciplinary connections	<ul style="list-style-type: none"> ✓ The content is related to other mathematical topics (connection of advanced mathematics with curricular mathematics and the connection between different mathematics content covered in the curriculum) or to the content of other disciplines, (an extra-mathematical context or rather links with other subjects from the same educational stage).
Social-professional practicality	<ul style="list-style-type: none"> ✓ The course content is useful for socio-professional insertion.
Didactical Innovation	<ul style="list-style-type: none"> ✓ Innovation based on reflexive research and practice (introduction of new content, technological resources, methods of evaluation, classroom organization, etc.)

For each Final Master's Thesis (TFM), the reasons given by their authors to justify the quality of innovation they proposed were analyzed by means of investigator triangulation with experts in the application of the OSA. This analysis essentially involved the selection of evidence showing the explicit or implicit use of some of the components and the characteristics of the different criteria of didactical suitability as proposed by the OSA.

The following section details the study of Professor Lopes' Final Master's Thesis, as an example of the analysis of this data.

PROFESSOR LOPES' DIDACTIC PROPOSAL

Professor Lopes' FMT (2014), entitled "A review of the introduction of Riemann's sums into High School Education", presents the design and the implementation of a didactic proposal for a group of third-year high school students to intuitively introduce the integral calculus through the study of the areas of 2D geometric forms. Lopes (2014) explains that it is possible to introduce methods and notions of the integral calculus in High School Education intuitively, starting with area-calculation problems for curvilinear shapes. That is, the aim is to broaden the calculation of areas habitually studied in Elementary Education through the study of area-calculation of curvilinear shapes, using both Archimedes' and Riemann's methods.

To be specific, Lopes' FMT (2014) is organized into four chapters; in the first, the professor presents, using literature reviews, the argument: "Should integral calculus be used in Elementary School?" Based on this question, the professor seeks to justify- through the study of literature- the use of two methods: Archimedes' method (used to calculate the area of a circle) and Riemann's method (used to calculate the area of three curvilinear shapes: circles, ellipsis and polynomial shapes with an x axis). In the second chapter, Lopes (2014) explains the didactic unit which was implemented with a group of third-year students from a state high school in Brazil. The group was formed of 41 students but at the beginning of the year, only 36 students attended the classes and participated willingly in the project. In this second chapter, the professor also explains in detail the initial self-evaluation he performed with the students and, in particular, he explains the method for evaluation previous knowledge on certain geometry topics, on mathematical software knowledge and also on the expectations of the project.

In the third chapter, professor Lopes describes the implementation he carried out. This section is a sequential report in which the author explains what happened during the implementation of the didactic sequence, placing emphasis on the set tasks, what the students learnt and the interactions made during the implementation. This is a written from the perspective of the professor but, in his very review, the professor ensures he presents evidence of the statements he makes. In the fourth and last chapter, the professor presents his reflections and conclusions on the implementation he carried out. In this way, it can be said that Lopes' proposal (2014) covers the four phases of didactic design (preliminary study, design, implementation and evaluation), which other models of mathematics teachers'

knowledge also cover, in order to answer the most fundamental question: “What knowledge should a mathematics teacher have to be able to appropriately manage their students’ learning?” (Pino-Fan, Assis & Castro, 2015).

In the following section, we present the analysis of the implementation Lopes carried out. We based our organization of the information on the basis of the didactical suitability notion.

PROFESSOR LOPES: ANALYSIS OF HIS OWN IMPLEMENTATION

When teachers have to reflect on a didactic proposal that implies a change to or an innovation in their own practices, they implicitly employ some of the *didactical suitability criteria*. Lopes’ FMT (2014) has also allowed us to deduce the use of some of these criteria in the justification and reflection on the suggested proposal. In the following subsections, we show the extent to which the author considered- implicitly and explicitly- the suitability criteria put forward by OSA in attempt to defend his didactic proposal as improvement for mathematics teaching.

Epistemic suitability

Errors and ambiguities

Lopes (2014) makes no comments relating to possible errors that could arise in his proposed instruction process. Further, he explains in his review that he carried out an initial evaluation and three other progressive evaluations without explicitly commenting on the students’ misunderstandings and/or difficulties that, in his opinion, were due to his explanations. It must be highlighted that the author of the FMT attributes difficulties and the lack of profound learning associated with Riemann’s method to lack of time and not to ambiguities or existing errors in his explanation. However, it must also be taken into account that the author approaches the notion of limit intuitively and the investigation into the teaching of this notion has brought to light that we are dealing with an ambiguous teaching style which generates difficulties and confusion for the students. That said, on the basis that we are talking about third-year high school students, this kind of approach is appropriately adapted to the target educational level. Thus, we can infer that the ambiguities associated with this intuitive approach are reasonable considering the targeted educational level.

Diversity of processes

Lopes (2014) justifies the ‘innovative’ nature of his proposal by pointing out that it encourages students to perform relevant mathematical processes, in particular that of mathematical modelling. In his own words, he explains:

“In this way, the application process, divided in three stages, aims to build knowledge through the use of mathematical models. Starting with the first construction, on the basis that the topic is studied in depth and new elements arise, other models are built based on the previous ones (...).” (Lopes, 2014, p. 22)

The professor also considered that his innovative proposal allows students to perform other relevant mathematical processes such as connections, meaningful constructions, problem-solving, etc. Lopes (2014, p. 21) states:

“In this sense, the aim is to introduce concepts of advanced Mathematics through area-calculation, to acknowledge the numerous applications that the study of geometry provides, to guide the student in the construction and identification of different geometrical figures, to provide the student with the geometrical and arithmetical construction of mathematical concepts and entities, to awaken the student’s creativity and enthusiasm to learn geometry, to create geometrical models with the students, making connections with reality, and to provide situational problems with a geometric focus (...).”

It is evident in his review that some of the processes mentioned were in fact developed during the implementation of his proposal. For example, Lopes (2014, p. 76) shows evidence that the students performed, respectively, problem-solving, argumentation and analogy techniques (on calculating the area of the eclipse and the circle):

- 1 T: How can we solve this problem?
- 2 T: Which of the two methods we studied would be more useful to solve this situational problem?
- 3 Ss: [Exchanging ideas, most of the students agree that the best method to solve the problem would be Riemann’s method. A group of students (Ss) explains:] it’s just a matter of dividing the base into equal parts and drawing rectangles, much in the same way as calculating the area of the circle and the eclipse.
- 4 T: [The teacher provides students with a printed copy of the graphical construction and guides them to divide the measurements of the base (horizontal axis), initially into ten equal parts and to calculate the approximate area.

Representation

In his thesis, the professor generally presents explicit reflections on the fact that his didactic proposal for teaching area-calculation is more representative (since it thoroughly explores the area-calculation of curvilinear figures) than the proposals that are commonly implemented at high school level.

Cognitive suitability

Regarding this type of suitability, in Lopes’ work (2014) there are comments, reflections, etc., that lead us to the conclusion that the author considers- mostly implicitly- the criteria of cognitive suitability.

Background knowledge

The professor carried out an initial evaluation in order to find out whether the students had the necessary background knowledge to study the set topic. That is, he made sure that the students were familiar with certain concepts and, more specifically, he dedicated part of the time intended for the implementation, to revising area-calculation of triangles and quadrilaterals, and to the study of trigonometric ratios. According to the teacher, the learning objectives were attained by the students, “and there is confirmation that the Archimedes and Riemann methods are in the students’ zone of proximal development” (Lopes, 2014, p. 19).

Adapting the curriculum to individuals’ different needs

With the narrative of the teacher can not to conclude if, he considers or not considers expansion or reinforcement activities. Nevertheless, when he assesses the learning related to the Riemann method, he concludes that many students will not in fact learn in this way and he argues: “(...) a slightly more extensive study period would be necessary in order to ask the students, during an evaluation activity, to interpret results more thoroughly, considering that each student is unique and, therefore, needs more or less time to learn” (Lopes, 2014, p. 92).

Regarding *learning*, the professor very clearly states that he must carry out evaluations to verify whether his innovative proposal facilitates students’ understanding of the material. Therefore, in addition to the initial evaluation, the professor carries out three progressive evaluations that show the acquisition of the implemented competences/learning. With these evaluations, the professor concludes that the students clearly did learn about the area-calculation of quadrilaterals and triangles. He also concluded that the same could not be said for those learning by means of the Riemann method, which he claims was due to lack of time.

In addition to the previously mentioned evaluations, the author of the FMT incorporated a final questionnaire to obtain the students’ own evaluation of the instruction process employed. One of the questions asked students to grade their level of understanding of material that was covered. The students’ answers confirmed the conclusions of the progressive evaluation, in the sense that the students learnt reasonably well (Lopes, 2014, p. 93-94):

- 1 Ss: It was considerably easy to understand, I managed to apply the material well.
- 2 Ss: The proposed material was clear and allowed understanding of the content, despite presenting some difficulties.

High cognitive demand

Lopes considers that his proposal entails high cognitive demand for the students, since it activates relevant cognitive processes. In fact, the high cognitive demand is the other side

of the coin of the diversity of processes mentioned in the criteria of epistemic suitability. In other words, when the teacher chooses a didactic proposal that implies the performance of relevant mathematical processes (good Mathematics), he/she is setting tasks for the students that require a higher cognitive demand.

Interactional suitability

Teacher-Student interaction

In his work, the professor describes substantial “teacher-student interaction”, in which the teacher asks questions and the students give answers, which according to him, “facilitates the students’ understanding” (Lopes, 2014, p. 32). The professor also presents some examples of how this type of interaction helps to clarify and solve doubts that the students might have.

Interaction among students

In his review, the professor also mentions that the students worked in small groups and concludes that this organization allowed students who did not usually participate in class to express themselves in a larger group. “It’s worth mentioning that a group of students who usually participated in problem-solving and activities without voicing their opinions in the large group spoke up on this occasion, saying that they were enjoying the activities and working as a group”

Autonomy

Based on the professor’s FMT it is possible to conclude that there were moments in which the students’ autonomy was encouraged. For instance, “the students had to do homework” (Lopes, 2014, p. 67) and it was also evident that there were moments in which the students took responsibility for their studies (carrying out exploration, formulation and validation).

Formative evaluation

As mentioned in the cognitive suitability section, the professor carried out a progressive evaluation that enabled a systematic observation of the students’ cognitive progress.

Mediational Suitability

Material resources

Both in the planning and in the implementation the instruction processes it is possible to observe the use of *material resources* such as manipulatives, calculators and computers. The professor explains that he used the *GeoGebra* software and calculators in his instruction processes. Regarding *GeoGebra*, he presents some implicit evaluative comments about the advantages of including this dynamic geometry software in the instruction process. According to Lopes (2014), “Archimedes continued inscribing and circumscribing polygons, always duplicating the number of sides of the previous polygon until inscribing and

circumscribing a polygon of 96 sides. We can see that, as the number of sides increases, the closer the area taken up by the polygon is to the area of the circle" (Lopes, 2014, p. 44-45).

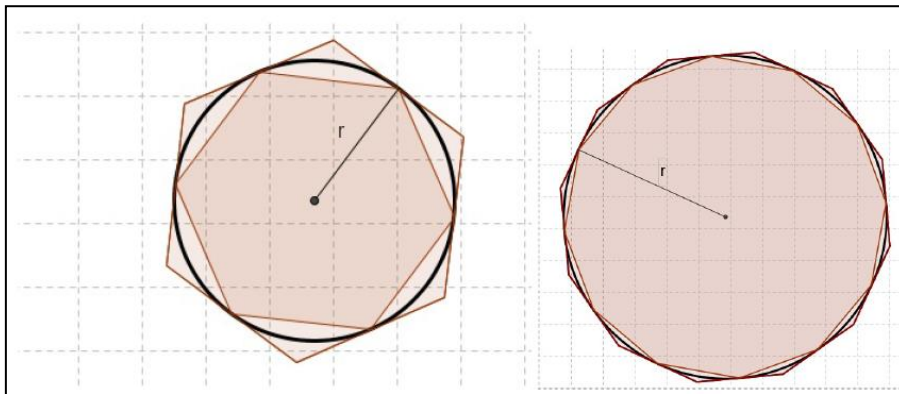


Figure 1. Inscription and circumscription of regular polygons in the single-line circle using the *GeoGebra* software.

Number of students, timetable and classroom conditions

The professor makes several comments on this point. Rather relevantly, he explains that the number of students and the classroom conditions (both the physical space as well as the computer laboratory) somehow defined the use of *GeoGebra*. Thus, the teacher to illustrate and demonstrate mathematical practices (e.g., the area- calculation of quadrilaterals and triangles) mainly used the *GeoGebra* software.

Time

Concerning *time -for collective teaching and for learning-*, the professor makes comments and assessments about three characteristics: adequacy of pretended meanings in the time available, the time devoted to the most important or essential topics and the time devoted to topics that proved more difficult for the students. In relation to the first characteristic, the professor very clearly states that he could not accommodate the intended sense to the time that was available. To be precise, he states that he did not have enough time to finish explaining what he had planned from the Riemann method. Regarding the second characteristic, the professor specifies that it took him a lot of time to ensure the required background knowledge and that, consequently, he did not have time to solve the initial contextualized problem with which he hoped to link the Archimedes and Riemann methods. Finally, regarding the third characteristic, it is inferred from the professor's comments that it was not possible to carry out the whole study due to lack of time (for instance; there was not enough time to explain the Riemann method in depth).

Affective suitability

Concerning this suitability criterion, no comments regarding the *interests and needs of the students*, nor comments about the students' *attitudes*, were found in Lopes' FMT (2014). Regarding *emotions*, the professor points out that the very implementation of the class promoted the students' self-esteem.

Ecological suitability

According to the criteria and objectives that the teachers had to consider when conducting their final Master's theses, the professor Lopes asserts that his proposal is a *didactic innovation* that can be adapted to the elementary school *curriculum* and, according to his students, it facilitates social and professional integration (*social-professional utility*) and that it presents an intra-mathematical *connection* to higher level Mathematics (*intra and interdisciplinary connections*), (Lopes, 2014, p. 94):

- Ss: It's important because with these formulas, we can work out the area of any object, and we could use them for the construction of houses, buildings, or important monuments.
- Ss: So, they're going to be useful in day-to-day life, though perhaps not every day, and I'll use them in some way in my profession.

Reflections on the case analysis

One advantage for the reader of this FMT is that the author is justifying the nature of a proposal which has already been implemented and thoroughly reviewed. The author implicitly uses all of the didactic suitability criteria. An important aspect to highlight is that this FMT clearly demonstrates the issue of finding a balance between each of the suitability criteria. On one hand, the author plans an innovation with high epistemic suitability and he demonstrates in his review that he also made a substantial effort to achieve high cognitive suitability. On the other hand, however, he also demonstrates that he was obliged to neglect part of the content he had planned; in particular, he could not solve the initial problem which was the very motive of his didactic proposal and the learning was not complete (in particular, the Riemann method) due to the fact that the mediational suitability was not adequate; to be precise, there was not enough time. The author's conclusion states that if, in future implementations, cognitive and epistemic suitability are not to be neglected, then it would be necessary to allow more time.

FINAL REFLECTIONS

Regarding the suitability criteria, the 29 teachers implicitly used in their arguments to justify that their FMT proposals lead to an improvement in mathematics teaching, Breda and Lima (2016) observed, overall, the use of epistemic and ecological suitability criteria and, to a lesser extent, the mediational suitability criteria. However, other criteria such as those of cognitive, emotional and interactional suitability were either not considered or were only

briefly touched on. Lastly, it must be highlighted that the teachers who implemented their didactic proposal in the classroom carried out a more detailed didactical analysis in comparison to those teachers who did not do so given that: i) they involved a greater number of criteria; ii) the components and characteristics of the suitability criteria that were employed, were inferred on the basis of arguments showing a high level of reflection; and iii) they were concerned with achieving a balance of the different criteria (epistemic, cognitive, affective, interactional, mediational and ecological). We believe that these findings, in this Naturalist Inquiry, confirm the third pattern. On the other hand, it can also be noted that positive assessment (implicit or explicit) of some of the characteristics of the didactical suitability criteria components is based on certain tendencies of mathematics teaching (for example, the use of technology resources is considered positive), thus also confirming the fourth pattern.

These findings support the conclusion reached by Ramos (2006) and Seckel (2016), in research conducted in different contexts to the explained in this paper, since they were not naturalistic studies (such studies were experiments of the development of the teacher's competences and knowledge). These authors believe that when teachers must reflect on a didactic proposal which implies a change to or innovation in their own practices, they implicitly employ some of the suitability criteria put forward by the OSA. The analyses of the 29 FMTs explored in this multiple case study of a Naturalist Inquiry style, also enable us to infer the implicit use of some of the characteristics of these criteria in the justification of the improvement represented by the implemented proposal. Further, the analysis of the FMTs demonstrates that the teachers' reflections go beyond spontaneous reflection, however, it is not suitably directed, amongst other reasons, due to the lack of an explanatory and broad guideline in the PROFMAT, which sets out to guide the teachers' didactical analysis.

The main limitation of this study is its extension. That is, the findings are closely related to a specific Masters (PROFMAT), offered in a specific region of Brazil (Rio Grande do Sul). As we explained in the last section of the theoretical framework, the studies carried out with different methodological designs, in different institutional contexts of initial and permanent training and in different countries (Spain, Chile, Peru, Mexico, Brazil and Ecuador) confirmed the same results; that is, the finding that when teachers reflect on their practice, they use -implicitly or explicitly- the suitability criteria proposed by the OSA. However, the same cannot be said for similar experiences in these countries, or others in fact (for example Turkey), on the basis that there are no studies to attest them. Nevertheless, when other investigations are developed in other contexts, we believe that the results would not be so far from those we have presented in this investigation.

One aspect which is difficult to explain is the reason why the criteria of didactic suitability function as implicit patterns in the teachers' discourse, when the teachers themselves have to evaluate instruction processes without specific training on the use of this analysis tool. Now, the answer to this very question is related to the origins of this construct. In the OSA, the didactic suitability criteria, its components and characteristics were

constructed on the basis that they should be constructs which rely on a certain amount of consensus within the Mathematics Education community, albeit the local one. As a result, it was considered that, given the ample consensus they generate, the principles of the NCTM (2000) could serve as the basis for some of the didactic suitability criteria, or rather, they could be considered as some of the components themselves. On the other hand, for the development of the didactic suitability construct, some of the contributions (principles, results, etc.) of the different approaches of the Mathematics Education area were also taken into account (Godino, 2013). Therefore, one of the plausible explanations that the suitability criteria can be considered as teachers' reflections patterns is related to the extensive consensus that they themselves generate amongst persons involved in Mathematics Education.

As a final thought, it is opportune to highlight that one way of making a more detailed reflection, which enables the improvement of mathematics teaching, involves the explicit use of the suitability criteria, as suggested by Pino-Fan, Godino and Font (2016), and as has been conducted in several research and training proposals (Godino, Batanero & Font, 2008; Giménez, Font & Venegas, 2013). Therefore, we consider that the use of the didactical suitability criteria in the teacher training processes could be a powerful methodological tool to promote and support teachers' reflection on their own practice (Breda & Lima, 2016).

To conclude, we wish to highlight that this article is a partial response to a question which is relevant to the training of mathematics teachers: Should the reflection on practice be guided? If so, with the use of what tools? Our response is that the reflection on practice should indeed be guided and that the use of the didactic suitability criteria (and more generally speaking, with the constructs of the DMKC model) can serve as useful tools for the teachers. We are looking at a result which coincides with other investigations which have highlighted: (1) that giving teachers the opportunity to reflect on their practice does not suffice, (2) that the teachers need tools to direct their attention to salient aspects of teaching episodes and (3) that these tools can be taught as part of teacher education (e.g., Turner, 2012; Star & Strickland, 2008; Sun & van Es, 2015; Giménez, Font & Vanegas, 2013; Seckel, 2016; Rubio, 2012; Nilssen, 2010).

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