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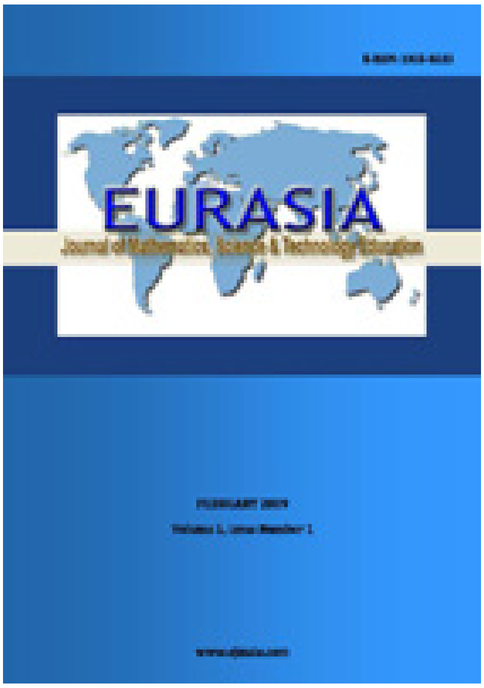


# Multi-Relation Strategy in Students' Use of a Representation for Proportional Reasoning

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# Multi-Relation Strategy in Students' Use of a Representation for Proportional Reasoning

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The purpose of this paper is to analyze in detail the process in which a 5th grade student learned proportion and percent using double number lines and to obtain insights into how his use of those representations supported his learning. A series of lessons were designed and conducted in a class and the whole learning process of one student was video-recorded. The analysis showed that he used double number lines to search for various multiplicative relationships in problem situations and their consistent pictures, rather than to make necessary numerical expressions systematically. The analysis also showed that such uses of number lines made it possible for him to control his proportional reasoning and develop his understanding of proportion and percent.

*Keywords*: cognitive tools, learning processes, elementary school mathematics, proportional reasoning, representation

## INTRODUCTION

Although mathematical knowledge of multiplicative structures, including proportions and percents, is a key for learning middle-school-level mathematics, it is often said that learning of proportions and percents is difficult for students. It is also well-known, however, that even students who have not yet learnt these mathematical ideas have rich informal strategies concerning proportions and they can solve problems which require the use of proportional reasoning (Lamon, 1993). If students have such informal knowledge about proportions, changing their informal knowledge into more formal one can be one of plausible approaches to teaching proportions and, in such an approach, diagrams and drawings are expected to be helpful for facilitating this transition (e.g. DeCorte et al., 1996). Because teachers' instructions in mathematics classes play a critical role for the development of students' learning of such complex domains as multiplicative structures (Behr et al., 1992), how the use of diagrams or drawings can promote students' learning of proportions and percents is useful information for the design of our teaching.

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In this study, the lessons of the unit of proportions and percents were designed and implemented, in which a certain type of diagrams was used to visualize students' proportional reasoning which they had before receiving these lessons. And the learning process of a 5th grade student was analyzed in detail to obtain an insights into processes of learning proportions and percents with that type of diagrams and factors which influence those learning processes. In the following sections, the previous researches about students' informal knowledge of proportions will be reviewed first. Then, the lessons and the learning process of the student who participated in those lessons will be described. Finally, through the analysis of the learning process and the changes in his use of the diagrams during the learning process, some features will be identified to obtain insights into students learning processes with such diagrams, as well as implications for instructions.

### Theoretical Background

After they receive formal instructions on proportions, students tend to use the rule of three and


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***State of the literature***

- Whereas it is often reported that students have difficulties in learning proportions and percent, there are some researchers who reported that students have rich informal knowledge about those concepts.
- Although the use of certain kinds of diagrams in learning proportions and percents has been recommended, there are little researches that analyzed the processes in which students use those diagrams to learn proportions and percents.

***Contribution of this paper to the literature***

- This paper gives the detailed analysis of the above process and shows that even elementary school students can develop conceptual understanding of proportions and percents by using dual number lines to make their use of proportional reasoning more conscious and flexible.
- This paper gives some features of students' use of dual number lines like the following: Instead of making numerical expressions systematically, the student attempted to search for multiple and consistent relations in problem situations and understand those situations.
- The analysis in this paper demonstrated some factors which might prevent students from using dual number lines smoothly:(a) studnets' number sense about decimal fractions; (b) the timing of plotting the base "1" in a double number line.

make equations based on it to solve problems about proportions (e.g. Avcu & Avcu, 2010). As mentioned above, however, even elementary school students who have received no formal instruction on proportions have informal knowledge about proportions and ratios and use it to solve problems whose inherent structures are multiplicative. Lamon (1993) implemented clinical interviews on proportions and ratios with 6th grade students before they had received any instruction in this domain. She found that the students used proportional and preproportional reasoning in solving problems involving proportions and ratios, and pointed out that "their conceptual and procedural competencies were greater than their symbolic competencies" (p. 59). She recommended that instructions should take advantage of those strengths and encourage students to extend their knowledge and strategies into the more complex domain of ratio and proportion.

Some researchers reported the fact that students also have rich informal knowledge about percents. Yoshida et al. (2000) asked the 5th grade students, who had not yet learnt proportions, what the 50%, 25%, 75% and 90% of 40 counters are respectively. Sixty-six percent of the students answered correctly to 50%, and about half of the students answered correctly to 25% and 75%. Furthermore, 40% of the students answered 36 or 35 as the 90% of 40 counters. Analyzing the students' strategies, Yoshida et al. (2000) found that the students used the informal strategies based on halving or the tenth of 40 counters. Similarly, Yamaguchi (2007) found that 5th grade students had a good sense of percents before they learnt percents in the mathematics lessons. He poured water into a cylinder bottle and stopped when the water reached at the one third of its height (cf. Moss, 2005). When he asked his students what percent we can call this water, many students answered "30 percent." He restarted pouring and asked the students to say "stop" when the water became 50% and 80% of the bottle. The students could say "stop" at almost appropriate moments in the both cases. They also knew that the water was called 100% when the bottle was filled with the water. Furthermore, when the water overflowed out of the bottle, the students called that situation 120% or 130%.

One of the factors which make learning of proportions and percents difficult is their compressed notations in which relational features of quantities are usually hidden. Dual number lines or comparison scales have been recommended to make those relational features explicit for students (Parker & Leinhardt, 1995). To provide a clear image of the proportional relationship of percent situations, Dole (2000) adopted dual number lines for 8th grade students. Through her analysis of students' notebooks, worksheets, diaries, and the ad hoc interviews with students, Dole (2000) found that "the number line assisted students in organizing the elements of percent situations, and students also referred to common percent benchmarks (such as 50% and 25%) to position values on the percent side of the number line and to estimate the problem solution" (p. 385). Because the organized elements on the dual number lines suggested visually and directly equations necessary to solve the percent problems, the students could make those equations easily with the help of the dual number line and they "quickly experienced success in solution attainment" (p. 385).

While Dole (2000) adopted dual number lines for 8th grade students and used their visual feature to help students make equations of the Rule of Three, van den Heuvel-Panhuizen (2003) developed the program for 5th and 6th graders learning percents, in which "the bar model emerges and evolves" (p. 18). In the program, the bar model was selected for 5th grade students and it was assumed that the bar model emerged when an occupation meter evolved from pictures which students draw to represent fullness. Fifth grade students can use the bar model to estimate required percents and calculate those percents based on the "1%-benchmark."

In the lessons for 6th grade students, it was assumed that the bar model was used for making sense of the results of the calculations and for investigating the problem situations to find answers. In the example in which students are required to find the original price of a camera when its 25%-discount price is 96 dollars, students can use the bar model or a dual number line to reinterpret the discount price as 75% or three-quarters of the original price and find the answer by adding one-third of 96 to the original price (p. 28).

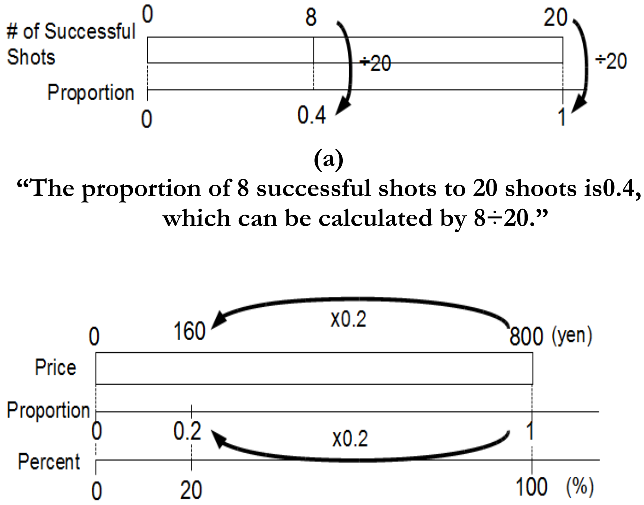
Shreyar et al. (2010) analyzed the lesson in which the students reflected on various ways of finding 20% of 69 dollars and 25% of 75 dollars and the bar model was used to make sense of those ways. The authors' analysis focused on the teacher's semiotic mediation of the conversation, and showed how the teacher directed the conversation so the meanings of percent and of the standard algorithm for calculating percents were realized and, through that realization, how a collective ZPD was enacted.

While these previous researches demonstrated the effect of students' use of dual number lines or recommended the use of the visual model to develop students' understanding of percents, they did not analyze the learning processes of students who participate in the lessons on proportions and percents and learn those mathematical concepts and skills using a certain visual model. But, because such analyses can provide us with useful information for designing lessons, it is necessary to analyze those learning processes. Thus, the purpose of this paper is to analyze the learning process of such a student in detail to obtain insights into how students learn proportions and percents with their use of a diagram like a dual number line.

### Gathering Data

The lessons on proportions and percents were designed and implemented in one 5th grade class in Japan. They consisted of nine 60-minute lessons and one 30-minute session in which the students were asked to solve various problems on proportions and percents. While these lessons were implemented as a part of regular curriculum, the teacher of these lessons was not the classroom teacher, but a graduate student who participated in our in-service training program. The worksheets developed by our team referring to the textbook were used instead of the usual textbook and the students were required to write their ideas on those worksheets instead of their notebooks. The students' worksheets were gathered and copied after each lesson, and were returned to the students at the beginning of the next lesson.

One video camera was settled at the back of the classroom to record the teacher's behaviors, the blackboard, and the students' behaviors in front of the blackboard. Another video camera was settled beside the blackboard to record the whole class. To record and analyze the students' use of the diagrams in detail, we selected five students by consulting with the classroom teacher and a graduate student recorded the learning of each student with his/her video camera throughout the 10 sessions. The copies of the worksheets and the video records of the selected students and the classrooms are the data used in the research.




**(a)**
**"The proportion of 8 successful shots to 20 shoots is0.4, which can be calculated by 8÷20."**

**(b)**
**"20% of 800 yen is 160 yen, which can be calculated by 800x0.2."**

**Figure 1. Basic Style of PM**

### Basic Characteristics of the Diagram

Throughout the lessons, our team used a certain representation which is similar to a dual number line (Figure 1). We called it "Proportion Meter" (PM) in the lessons. Unlike the usual dual number line, upper side consisted of a bar rather than a line. Since the upper side usually represents extensive quantities like areas, number of peoples, and prices, a bar was used so that students easily feel a kind of "extension." We drew arrows between numbers and wrote operations like "÷20" or "x0.75" beside those arrows in order to highlight multiplicative relationships between those numbers.

The use of this representation was based on two assumptions. First, during these lessons, we took the standpoint that the mathematical concept of proportion is a kind of quantity which represents to what extent a quantity in question (a quantity to be compared) is in comparison with another quantity (a base quantity). While we view a base quantity as 1 when expressing a proportion using a decimal fraction, we view it as 100 when using percent. PM was expected to show "to what extent" with its relative relationship between the lengths representing a quantity to be compared and a base quantity. Because it is said that students tend to pay attention to part-part relations rather than part-whole relations (Boyer et al., 2008; Singer & Resnick, 1992), the teacher often adopted a dynamic presentation of PM. That is, the teacher presented PM as follows: (i) At the first, empty PM was presented; (ii) the teacher gradually withdrew a red ribbon from the slit made at the zero-position of the bar of PM (Figure 2); (iii) the teacher asked the students to say "Stop" when the front edge of the ribbon reached at the position of the quantity to be compared; (iv) the numbers and arrows were added to PM one by one, basically visualizing the students' ideas (see Figure 1). This dynamic presentation was also expected to have the effect similar to shading grids to represent percents (Bennett & Nelson, 1994). In order that the students kept the idea that proportions are a kind of quantity for representing "to what extent," the teacher asked them to express proportions or percents as, for example, "good many," "so-so," or "very little" and to relate them to familiar benchmarks (0 as empty, 1 or 100% as full, and 0.5 or 50% as half).

Second, PM was adopted to make the students more conscious of their use of proportional reasoning. It is assumed here that when they are more conscious of their use of proportional reasoning, students can use it more flexibly and apply it to complex situations. In other words, PM was expected to help student control their use of proportional reasoning and extend its possibility as a cognitive tool (Nunokawa, 2008).

As mentioned above, van den Heuvel-Panhuizen (2003) presented a program for learning proportions, in which the representations changed from pictorial ones to more abstract ones and dual number lines emerged through this transition. Because the students in this research experienced a similar transition when they were at 3rd and 4th grades (Nunokawa, 2007) and they had also used dual number lines in learning multiplications and divisions of decimal fractions (cf. Hitotsumatsu et al., 2005), we decided to introduce PM in the first lesson to make their knowledge of proportional reasoning visible and explicit.

**Figure 2. Withdrawing a Ribbon**

## Takuya's Use of the Proportion Meter

In this section, I will focus on one student, Takuya (pseudonym), and his uses of PM at some stages of the learning of proportion. While Takuya came to take advantage of PM through the teacher's support and be able to use proportional reasoning more flexibly, the ways he used PM were not standard ones and are suggestive for us. That is why Takuya was chosen for the detailed analysis here.

### *Lesson 1: Expressing Skillfulness of Basketball Shots*

At the beginning of Lesson 1, the teacher presented the situation in which 4 basketball players made 20 shots each and the numbers of their successful shots were 20, 10, 5, and 2 respectively. He posed the today's task: how to express the degree of skillfulness of each player. First, the teacher introduced a proportion-meter (PM) and demonstrated dynamically how to represent 20 successful shots on PM. The teacher also introduced a rule that when all the 20 shots were successful, its degree of skillfulness was expressed as "1." To think about the degree of skillfulness of 10 successes, the teacher moved a red ribbon on PM and asked the class to say "Stop" when the ribbon came to represent 10 successes. The class could say "stop" at the appropriate timing. Seeing this PM, the students expressed the degree of skillfulness of 10 successes as "0.5." They explained that the degree should be a half of that of 20 successes because the number of successful shots became half. The teacher visualized the students' idea on PM (Figure 3). Next, the class thought about the degrees of skillfulness of 5 successes and 2 successes in the same way.

Takuya could find the degree of skillfulness of 5 successes to be 0.25 using proportional reasoning. But he calculated 0.25÷2 and 0.125÷2 in finding the degree of 2 successes and could not find the answer.

Finally, the class discussed the degree of skillfulness of 8 successes. Some students presented the following ideas and the students agreed that its degree was 0.4: (a) 0.1x4 because the degree of 2 successes was 0.1 and 2x4=8; (b) 0.5÷1.25 because the degree of 10 successes was 0.5 and 10÷1.25=8; (c) 1÷2.5 because the degree of 20 successes was 1 and 20÷2.5=8; (d) 0.5-0.1 because the degrees of 10 and 2 successes were 0.5 and 0.1 respectively.

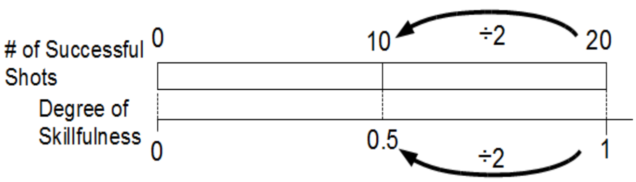



**Figure 3. The Degree of 10 Successful Shots**

Takuya calculated 20÷8=2.5 and mentioned the proportional reasoning which he used to find the degree of 2 successes, 1÷10. But he did not use 20÷8=2.5 to determine the degree of 8 successes. After he wrote "0.1" as the degree of 2 successes and "0.25" as that of 5 successes in his PM, Takuya said that the degrees of 4- and 3-successes were "0.2" and "0.15" respectively. Furthermore, he added to his PM "0.3", "0.35", "0.4", and "0.45" as the degrees of 6-, 7-, 8-, and 9-successes.

***Lesson 2. Introduction of the concept of proportion***

At the beginning of Lesson 2, the teacher defined the concept of proportion as follows: the number which represents to what extent the quantity to be compared is when the base quantity is expressed as 1. The students also understood that the degrees of skillfulness discussed in Lesson 1 were proportions of the successful shots when the total, 20 shots, was considered the base quantity. Then, the teacher asked the students to find a way of figuring out proportions of the successful shots without horizontal proportional-reasoning they used in Lesson 1 (Figure 3). The PM like Figure 4(a) was printed on the worksheet handed to the students.

Takuya added an arc between 5 and 8 to this PM , wrote "3" beside it, and wrote “3” between 2 and 5. He also added an arc between 0.1 and 0.25 and tried to calculated 0.25÷3. Then, he calculated 0.4÷0.25, added an arc between 0.25 and 0.5, and wrote "x2" beside that arc (Figure 4(b))[1]. Listening to the conversation between the teacher and his peer, he pointed to “20” and “1” at the right end of his PM. When he calculated 20÷2, the teacher began a class discussion.

In the class discussion, some students presented the idea that dividing the number of successful shots by the total led to the proportion of that number. The class checked that this rule was applicable to all the results they had found in Lesson 1. The teacher explained that division by 20 was valid because we viewed the total shots as 1 in finding these proportions. He presented the formula of proportions, (a proportion) = (a quantity to be compared) ÷ (a base quantity). When his peers presented the above idea, Takuya added "÷20" between each number of successful shots and its proportion in his PM.

In solving the exercise problems, Takuya wrote numerical expressions first and then added the numbers to his PM. The numbers were written at the appropriate positions. But he did not add arrows to represent the relationships between numbers. Only when he solved the last problem which asked the proportion of 117 passengers to the riding capacity of 130, he added vertical arrows between the numbers (Figure 5). But, even in this case, he wrote the numerical expression 117÷130 before writing the given numbers in PM.

***Lesson 3. Comparison of Juices using Proportions***

In Lesson 3, the following problem was posed to the students: "One hundred milliliter of Juice A includes 80mL concentrated juice, and 40mL of Juice B includes 20mL concentrated juice. Which juice is thicker?" Even though the students had learnt to express degrees of something using proportions in the last two lessons, there were some students who thought that two juices had the same taste based on the additive strategy. Takuya was one of such students. He thought that the two juices had the same taste and wrote that "100-20=80, 40-20=20, they had the same difference." Seeing what his peer wrote, he changed his answer into "Juice A is thicker" and wrote "80÷100=0.8." After calculating 40÷20=2 and 20÷40=0.5, Takuya also wrote "20÷40=0.5" on his worksheet. Furthermore, he wrote, "0.8-0.5=0.3, Juice A is 0.3mL more." In the class discussion, one student stated that the concentrated juice in Juice A is more than half while that of Juice B is just half. Based on this idea, the class accepted that it was appropriate to compare two things using proportions in such cases. The class made PMs of these juices to check this idea.

Third juice, Juice C, was introduced as follows: 40mL of Juice C includes 30mL concentrated juice. The teacher asked which was thicker, Juice A or C? Takuya

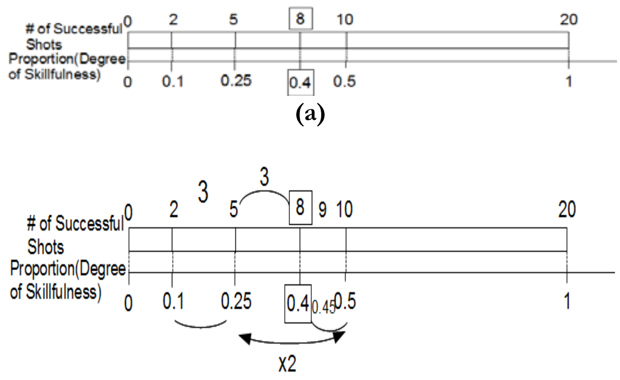



**Figure 4. PM for the skillfulness task and Takuya’s writing.**

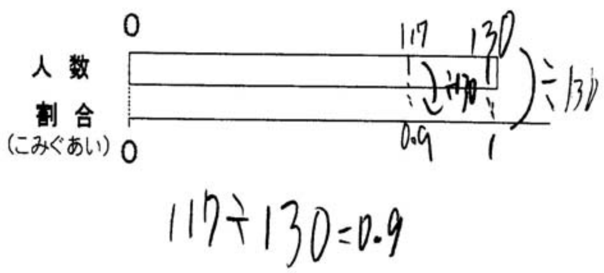



**Figure 5. Takuya’s PM of the occupancy rate Problem.**

added the given numbers and arrows to PM (Figure 6) and then wrote 30÷40=0.75. Although he made a correct decision that Juice A was thicker, Takuya wrote that "0.8-0.75=0.05, Juice A is 0.05mL more."

The next problem was concerned with a class which consisted of 16 boys and 20 girls. The students were required to find the proportion of the boys to the girls and the proportion of the girls to the boys. The latter was the first time for the students to find a proportion greater than 1. In finding the first proportion, Takuya added the given numbers and arrows to PM (Figure 7) and wrote 16÷20=0.8. When finding the second proportion, he talked with his peer and wrote a numerical expression. While he wrote the given and calculated numbers on PM afterward, he added no arrows to it.

### *Lesson 4. Introduction of Percents*

In Lesson 4, the concept of percent was introduced using the extended version of PM (Figure 8). Takuya spontaneously used PMs throughout this lesson. Takuya completed PMs first and then wrote correct numerical expressions (Figure 9). In making PM, he paid attention to the positions of numbers. When finding the percent of 63 autos to 140 vehicles, Takuya modified the position of "63" when he noticed that 63 is less than a half of 140.

### *Lesson 5 and the Beginning of Lesson 6: Finding the Quantity to Be Compared*

In Lesson 5, the students were asked to find 60% of 24m² using PM. This was the first time they tried to find out the quantity to be compared from the given information about the base quantity and percent. Takuya wrote "24" at the right end of the Area bar of PM (Figure 10(a)). After thinking a while, he wrote 0.6 and 60 at appropriate places. Takuya calculated 24x0.6 and found 14.4, but he also calculated 60÷24 and 0.6÷24. Then he calculated 14.4÷24=0.6, added an arrow from 14.4 to 0.6 to his PM, and wrote "÷24" beside this arrow. He also added an arrow from 24 to 1 to his PM and wrote "24" beside it. He added an arrow from 0.6 and 60, wrote "x100" beside it, and wrote "24x0.6=14.4, 14.4m²." He added "÷" sign to "24" beside the arrow from 24 to 1. After a while, he said "Ah," calculated 0.6x24, and wrote "Because 1x24=24, I calculate 0.6x24=14.4. Then I can also get 14.4÷24=0.6." When the teacher directed his attention to the directions of the arrows in his PM, he reversed the arrows and changed "÷24" into "x24" (Figure 10(b)).

Since no student mentioned in the class discussion the relationship between 60% and 100%, or 100x0.6=60, the teacher asked the students to explore the relationship between 100% and 60%. After calculating 100÷60 and 24÷100, Takuya added a horizontal arc between 60 and 100 in his PM and wrote "x0.6" under this arc. He calculated 60÷100=0.6 and wrote "100x0.6=60" (Figure 10(b)).

At the beginning of Lesson 6, the following problem was posed: "If 5% of lottery tickets are winning ones and this lottery has 80 tickets, how many of them are winning tickets?" Takuya wrote numbers "80", "100",

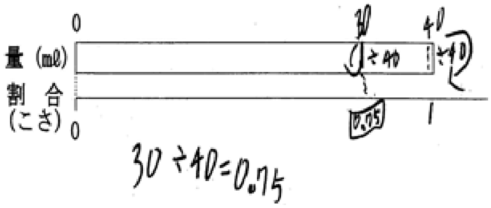



**Figure 6. Takuya's PM of the second Juice task**

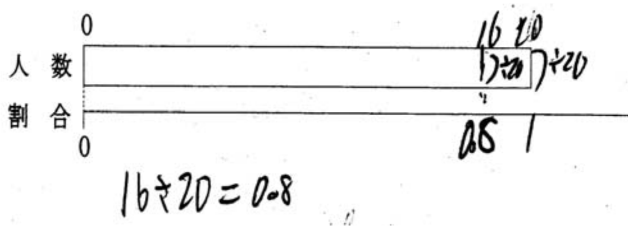
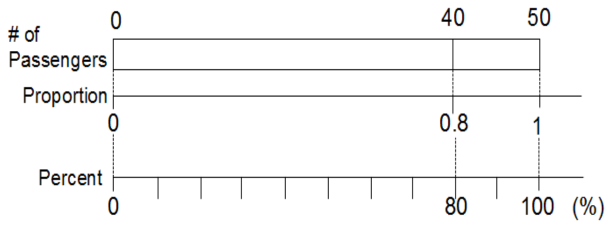



**Figure 7. Takuya's PM for the boys-girls problem**




(representing that proportion of 40 passengers to the riding capacity of 50 is 0.8, and its percent is 80%)

**Figure 8. Extended version of PM for learning percents**

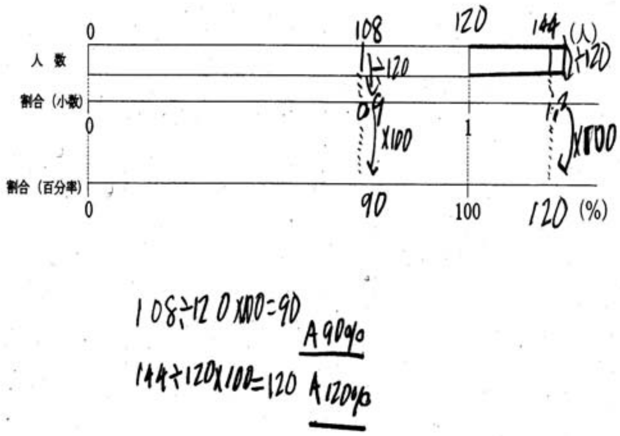
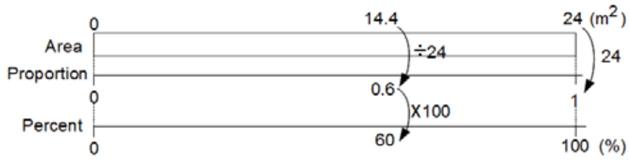
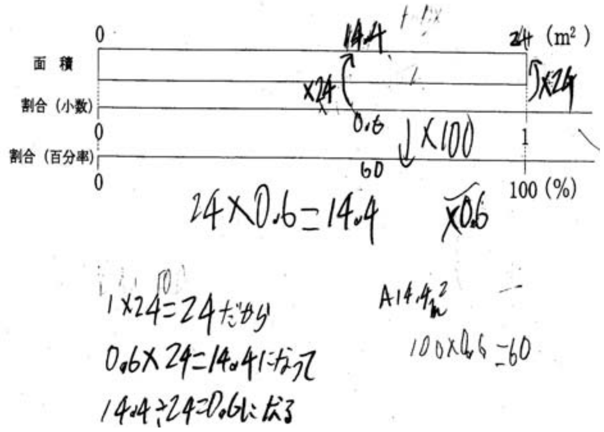


**Figure 9. Takuya's use of PM for a percent problem**




**(a)**




**(b)**

**Figure 10. Takuya's use of PM for finding the quantity to be compared**

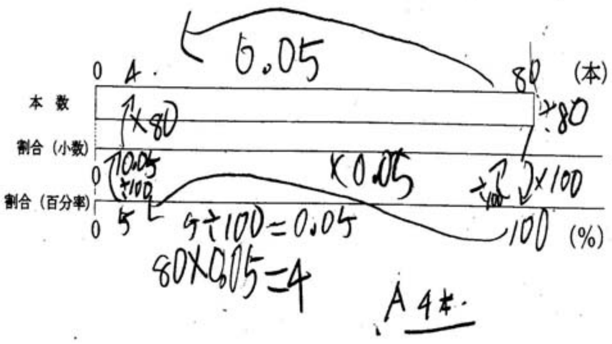



**Figure 11. Takuya's PM of the Lottery Problem**

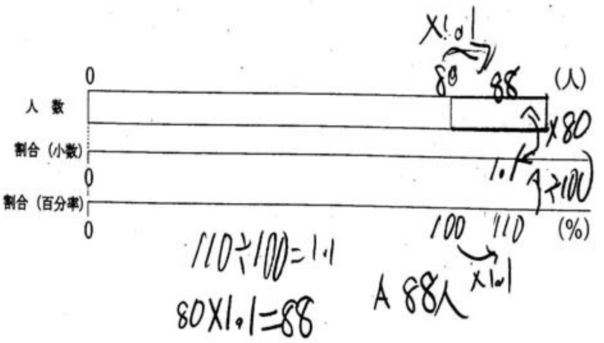



**Figure 12. Takuya's PM of the Railroad Car Problem**

"5", and "0.05" at the appropriate positions of his PM. He added a vertical arrow from 5 to 0.05 and wrote "÷100" beside that. He calculated 80x5 and 80x0.05, wrote "4" at the appropriate place, and wrote "80x0.05=4" on his worksheet. After a while, Takuya added a vertical arrow from 0.05 to 4 and wrote "x80" beside that. He also added "1" and wrote "÷80" between "80" and "1." Moreover, during the class discussion, he added a horizontal arrow from 80 to 4 and an arrow from 100 to 5 and wrote "x0.05" beside these arrows (Figure 11).

When he found the number of passengers of a railroad car whose riding capacity is 80 and whose occupancy rate is 110%, Takuya wrote "80", "110", "100", and "1.1" at the appropriate positions of his PM. Then he calculated 80x110 and 80x1.1 and added "88" to his PM. After he wrote "80x1.1=88" on his worksheet, he added a vertical arrow between 88 and 1.1 and wrote "x80" beside that. During the class discussion, he added a horizontal arrow from 80 to 88 and an arrow from 100 to 110 and wrote "x1.1" beside these arrows (Figure 12). He added these arrows before the teacher drew them on the blackboard.

#### *The Rest of Lesson 6: Finding the Base Quantity*

In the second half of Lesson 6, the students tackled the following problem: "The area of a flower garden is 90m². This is 30% of the whole garden. What is the area of the whole garden?" This was the first time for the students to find out the base quantity from the information about the quantity to be compared and percent. Takuya wrote "90" at the right end of the area bar of PM. When he wrote "30" at the appropriate position of the percent line of PM, he changed the position of "90" to more appropriate one (see Figure 13). He also wrote "0.3" at the appropriate position and a small box at the right end of the area bar. He added a horizontal arrow from 90 to this small box and wrote "x0.3" beside this arrow. Takuya calculated 90x0.3 using a calculator. He began to write something in this box, but he stopped writing. Seeing what his peer wrote, he calculated 90÷0.3 this time and wrote "300" in the small box. He changed "x0.3" beside the arrow into "÷0.3," and he wrote "90÷0.3=300" on his worksheet. Finally, he also added a horizontal arrow from 30 to 100 of PM and wrote only "0.3" beside it.

When finding the total number of lottery tickets whose 30 winning tickets are 15% of the whole tickets, Takuya wrote "30" at the right end of the number-of-tickets bar of PM although he could write "100" and "1" at the appropriate positions. After a while, he changed the position of "30" to the appropriate one and wrote "15" and "0.15" at the appropriate positions. He added a horizontal arrow from 30 to the right end, wrote "÷0.15" beside that, and calculated 30÷0.15. Finally, he added a horizontal arrow from 15 to 100 and wrote "÷0.15" beside it.

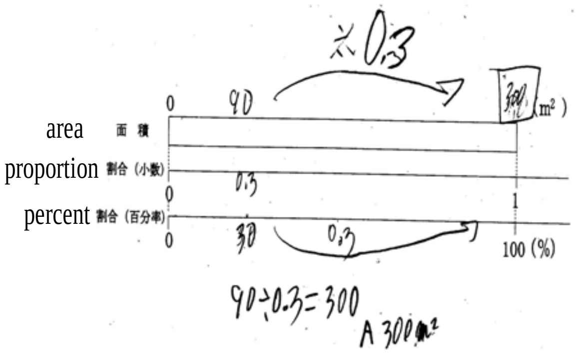



**Figure 13. Takuya's PM of the Garden Problem**

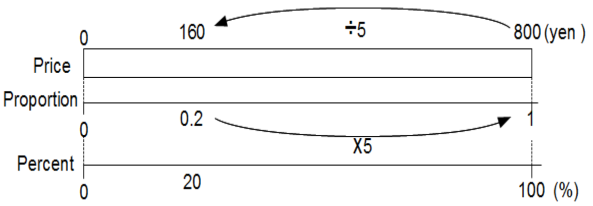



**Figure 14. Takuya's initial PM of Problem 2 at Lesson 7**

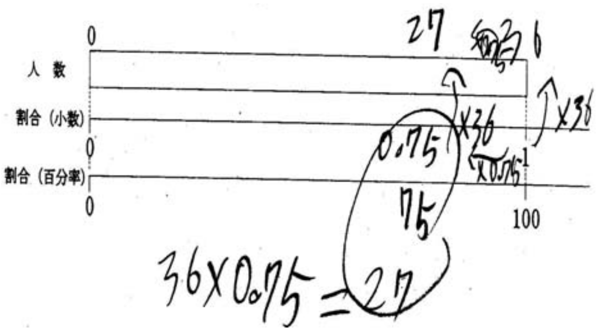



**Figure 15. Takuya's PM of Problem 4 at Lesson 7 with horizontal arrows**

When finding the riding capacity of a railroad car which carries 102 passengers and whose occupancy rate is 120%, Takuya wrote "120," "100," "1.2," and "1" at the appropriate positions of PM. But he wrote the number of passengers "102" above "100%" instead of "120%." After he calculated 102÷1.2=85 twice, Takuya changed the position of "102" to the appropriate position and wrote "85" above "100%." He added a horizontal arrow from 102 to 85 and an arrow from 120 to 100 and wrote "÷1.2" beside these arrows. Finally, he checked that 120÷1.2 became 100.

***Lesson 7: Five Exercises and Change-by-Percent Problems***

In the first half of Lesson 7, the students solved 5 problems about percent.

Problem 1: "find the area of the land, 8% of which is 24m$^2$"

Takuya wrote "24" and "8%" at the appropriate positions of PM, but calculated 24÷8=3 and 24x8=192, and wrote "192" above "100%." He wrote "0.08" at the appropriate position of PM and then calculated 24x0.08 and 24÷192. After he had solved Problem 2, Takuya returned to Problem 1, calculated 24x0.08 and 24÷0.08, and wrote "24÷0.08=300" on his worksheet.

Problem 2: "find 20% of 800yen"

Takuya wrote "800," "20," and "0.2" at the appropriate positions of PM. After calculating 800÷0.2 twice, he looked at PM and calculated 0.2x800=160 this time. He added "160" at the appropriate position and wrote "800x0.2=160" on his worksheet. After he had solved Problem 3, he returned to Problem 2, added a horizontal arrow from 0.2 to 1, wrote "x5" beside that, and then calculated 160x5. He also added another horizontal arrow from 800 to 160 and wrote "÷5" beside that arrow (Figure 14).

Problem 3: "find the percent of 72 girls to a total of 160 students"

Takuya wrote "160" at the appropriate position of PM. He wrote "72" at two thirds from the left end of the bar of PM. He moved this "72" to the middle of the bar. After calculating 160÷72 and 72÷160, he changed it again to the appropriate position, a little left from the middle. Takuya wrote "0.45" and "45" at the appropriate positions and wrote "72÷160=0.45, 45%" on his worksheet. After he drew a horizontal line from 160 to 72, he calculated 0.45x4, 0.45x2, and 0.45x2.5. While he wrote nothing beside this line, Takuya added a vertical arrow from 160 to 1 and wrote "÷160" beside that. He added another vertical arrow from 72 to 0.45 and wrote "÷160" beside it.

Problem 4: "find 75% of 36 persons"

Takuya wrote "36," "75," and "0.75" at the appropriate positions of PM. He calculated 36x0.75 and wrote "27" at the appropriate position. After he wrote "36x0.75=27, 27 persons" on his worksheet, he added vertical arrows from 1 to 36 and from 0.75 to 27 and wrote "x36" beside those arrows (see Figure 15).

Problem 5: "find the amount, 24% of which is 48dL"

Takuya wrote "24," "48," and "0.24" at the appropriate positions. After he calculated 0.24x5, 0.24x4.5, 48x0.24, and 48÷0.24, he wrote "200" at the appropriate position. He added a vertical arrow from 0.24 to 48. When talking with the teacher, he deleted this arrow, drew a horizontal arrow from 48 to 200 and an arrow from 0.24 to 1, and wrote "÷0.24" beside the latter arrow.

During the class discussion, Takuya made some modifications to his PMs. He newly added a horizontal arrow from 0.08 to1 to the PM of Problem 1. To the PM of Problem 4, he added a horizontal arrow from 1 to 0.75 and an arrow from 36 to 27 and wrote "x0.75" beside those arrows (Figure 15). Concerning the PM of Problem 2 (Figure 14), he changed "x5" and "÷5" beside horizontal arrows into "x0.2."

In the second half of Lesson 7, two “change by percent” problems (Parker & Leinhardt, 1995) were used and the students were asked to find the following prices: (a) before-tax price is 500yen and sales tax rate is 5%; (b) 20% off of 1500yen shirt. When solving (a), Takuya wrote "500," "5," and "0.05" at the appropriate positions of PM. He started to write something at the price corresponding to 5%. But he stopped writing and calculated 500÷0.05 and 500x0.05. He wrote "25" at the appropriate position of PM. He added a horizontal arrow from 1 to 0.05 and an arrow from 500 to 25 and wrote "x0.05" beside those arrows.

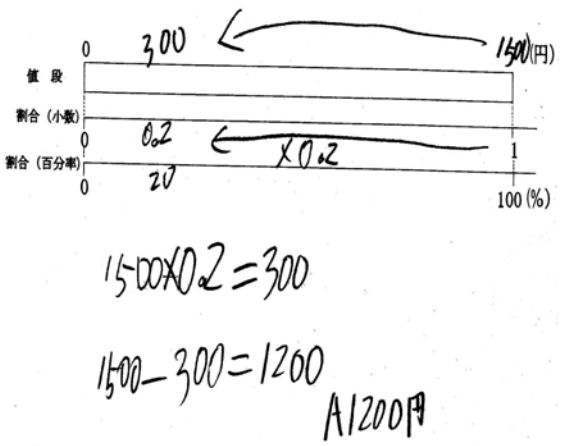



**Figure 16. Takuya’s PM of the Discount Problem**

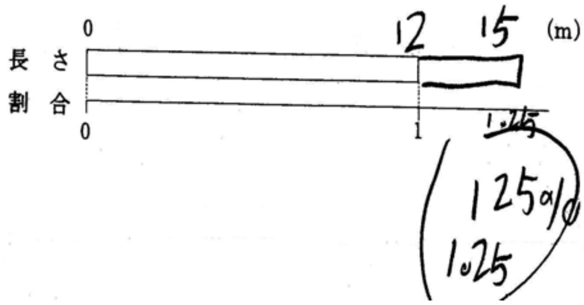



**Figure 17. Takuya’s PM for a Proportion of 15m to 12m**


**Figure 18. Takuya’s Drawing for the Book Problem**

When solving (b), Takuya wrote "1500," "20," and "0.2" at the appropriate positions of PM. He added a horizontal arrow from 1 to 0.2, wrote "x0.2" beside it, and then calculated 1500x0.2. He wrote "300" at the appropriate position of PM. He also added a horizontal arrow from 1500 to 300 (Figure 16). Finally, Takuya wrote on his worksheet, "As 0.2 is 1x0.2, I calculated 1500x0.2 and got 300. Then I calculated 1500-300 and got 1200 as the answer."

### *Lesson 8 and 9: Learning Bar Graphs and Pie Charts*

In Lesson 8, the teacher introduced bar graphs. The students were required to read a percent of each item from a given bar graph and find the number of that item using the percent they had read. In solving this problem, Takuya smoothly calculated the number of each item, like 50x0.42=21. The next problem required the students to find out the percent of each item in a table and make a bar graph. Takuya performed wrong calculations at first. He did not construct PM and calculated 23÷11 and 23x11 instead of 11÷23 when figuring out the percent of 11 persons to the total of 23 persons.

In Lesson 9, the teacher introduced pie charts. Takuya's behaviors were similar to those in Lesson 8. When finding the percent of each item in a table, he made similar mistakes and calculated, for example, 850÷250 instead of 250÷850 when figuring out the percent of 250 people to the total of 850 people.

### Lesson 10: Solving Various Problems

In Lesson 10, the students solved various problems about proportions and percents. When he found a proportion of 12m to 15m and a proportion of 15m to 12m, Takuya wrote the given numbers at the appropriate positions of PMs (Figure 17) and found the right proportions.

To the problem which asked a proportion of 3 absentees to 24 students in a class, Takuya calculated 3÷24=0.125. Even after he wrote "13%" as the answer, he also calculated 24÷3, 24x0.13, 24÷0.13, 3x24, and 0.13÷24. He looked at a peer calculating 3÷24=0.125 and changed "13%" into "12.5%." In finding a proportion of attendees to the class, he calculated 24÷21. Also in this case, he looked at a peer and modified that calculation into "21÷24." When finding a percent of a tax-inclusive price (630yen) to its price without tax included (600yen), Takuya calculated 600÷630. Only after checking the previous worksheet, he changed it into 630÷600.

Takuya found 4% of 300 eggs by calculating 300x0.04. He could decide correctly which lottery is easier to win, 16 winning tickets out of 40 or 7 winning tickets out of 20, referring to proportions. When he solved the increment problem ("There were 125 students in a school last year and the enrollment increased by 10 students this year. Find the percent of the number of this-year students to that of last-year students."), he calculated 10÷125 and 125÷10 instead of 135÷125. He reread the problem statement, but he calculated 115÷125.

The last problem was the following one: "You have read 48 pages of a certain book and the remaining is 60% of the whole book. How many pages does this book have?" Takuya calculated 0.6x48 and 60x48 and wrote "2880" in the answer space. He seemed confused when he calculated 48÷2880. Then he calculated 60÷48=1.25 and 48÷125. After a while, he drew a diagram like Figure 18 and moved his pencil between 48 and a small box like a horizontal arrow. The lesson ended when he thought about this problem.

## Features of Takuya's Use of PM and His Learning

### *Development of Takuya's Use of Proportional Reasoning*

Although he could use proportional reasoning in the simple cases even in Lesson 1, Takuya's reasoning about multiplicative-structure situations showed some weaknesses: (a) being limited to halving; (b) lack of control of proportional reasoning; (c) influence of additive strategy.

#### *(a) Being limited to halving*

He found the degree of 5 successes to be 0.25 by halving, 1÷2 and 0.5÷2. When finding the skillfulness of 2 successes after that, he used this 0.25 and calculated 0.25÷2 and 0.125÷2 without examining whether halving was appropriate also in this case. His use of proportional reasoning was tightly associated with halving just like the students reported in the other works (Misailidou & Williams, 2003; Pothier & Sawada, 1983).

#### *(b) Lack of control of proportional reasoning*

When finding the skillfulness of 8 successes, Takuya calculated 20÷8=2.5, added an arc between 20 and 8 on his PM, and wrote "2.5" beside that arc. But he could not take advantage of this result, for example, by calculating 1÷2.5.

#### *(c) Influence of additive strategy*

After he failed to use the result of 20÷8=2.5 to find the skillfulness of 8 successes, Takuya wrote "1] 0.05" on his worksheet. This meant that the skillfulness of 1 success was 0.05. He found the degrees of skillfulness of other cases using build-up strategy. He found the skillfulness of 4 successes by subtracting 0.05 from that of 5 successes (0.25-0.05=0.2), and found the skillfulness of 3 successes by subtracting 0.05 again from that of 4 successes. He found the degree of 6 successes by adding 0.05 to that of 5 successes (0.25+0.05=0.3) and figured out the degree of 7, 8 and 9 successes in the same manner.

While, as shown above, his use of reasoning was very naïve one in Lesson 1, Takuya became able to use proportional reasoning intentionally at the end of Lesson 7 and his use of PM was helpful for such intentional uses of proportional reasoning. When finding the 20%-off price of 1500yen shirt in Lesson 7, Takuya added an arrow from 1 to 0.2 in PM and wrote "x0.2" beside it. Based on this PM, he calculated 1500x0.2 to find 20% of 1500yen. Finally, he wrote on his worksheet as follows: "As 0.2 is 1x0.2, I calculated 1500x0.2 and got 300. Then I calculated 1500-300 and got 1200 as the answer." This implied that he was conscious of his use of proportional reasoning here and made the numerical expression based on this proportional reasoning. Furthermore, he could successfully combine the proportional reasoning (1500x0.2) and the additive reasoning (1500-300) in his solution. That is, he could use proportional reasoning intentionally enough with the help of PM and his use of proportional reasoning on PM directly supported his choice of the operation.

His intentional use of proportional reasoning was supported by his use of PM. When he utilized PMs in Lesson 10, Takuya could solve a little complicated problem in which the dividend was larger than the divisor (e.g. a proportion of 15m to 12m). But he made incorrect numerical expressions and could not correct them based on his understanding of problem situations when solving a similar type of problems without using PMs. For example, when finding a percent of the tax-inclusive price (630yen) to the price without tax included (600yen), Takuya performed an incorrect calculation 600÷630. He corrected this calculation by checking the previous worksheets, not by examining the multiplicative relationships in the problem situation. When solving Absentees-and-Attendees Problem and Enrolled-Students Problem in Lesson 10, Takuya did not use PM and could not determine appropriate numerical expressions with confidence. When solving the last Book Problem, he did not make progress without PM. When he finally started to use PM, he did not have enough time for completing it.

As discussed above, Takuya demonstrated development in his use of proportional reasoning and this development was supported by his use of PM.

### ***Changes in Takuya's Use of Proportional Reasoning***

Because PM was used to record the students' proportional reasoning in Lesson 1 and 2, the weaknesses which were observed in Lesson 1 remained in Takuya's thinking in Lesson 2. Although he used the results concerning 5 and 8 successes and calculated 0.4÷0.25=1.6, Takuya failed to interpret the relationship between 5 and 8 multiplicatively and paid attention to their difference 3. Moreover, he wrote "x2" beside the arc between 0.25 and 0.5, but he did not write "x1.6" beside the arc between 0.25 and 0.4. This suggested that he could not control his use of proportional reasoning other than halving and did not use PM to clarify relations between numbers. Similarly, Takuya interpreted the relation between 2 and 5 in terms of their difference, 3, and calculated 0.25÷3 to find the skillfulness of 2 successes using that of 5 successes.

After he learned the formula for finding proportions in Lesson 2, Takuya figured out proportions using this formula and wrote the given and the calculated numbers in PM afterward. Even though such descriptive role of PM was useful for checking his solutions, PM did not yet function as a thinking tool for controlling or managing his use of proportional reasoning.

Takuya began to use PM mainly to record his solutions in the second half of Lesson 2. But he still adopted additive reasoning when solving Juice problems in Lesson 3. When comparing the tastes of Juice A and B, Takuya made his first decision based on the differences between the whole juice and concentrated juice. Furthermore, even after comparing them using proportions, he calculated the difference between two proportions and wrote "Juice A is 0.3mL more." This failure seemed to lead him to using PM to comprehend a multiplicative relation when comparing Juice A and C. This is the first time for Takuya to use PM to comprehend a multiplicative relation and make a numerical expression. After this experience, he used PM both for recording his solutions and for comprehending multiplicative relations in Lesson 3 and 4. He seemed to use PMs to comprehend multiplicative relations when he solved new types of problems: Boys-and-Girls problem used in Lesson 3 was the first problem which was not about a part-whole relation but about a part-part relation; Exercises in Lesson 4 were problems about new topic, percents.

These two usages were combined when he found 60% of 24m$^2$ in Lesson 5. Although he calculated 24x0.6=14.4 and 14.4÷24=0.6, he searched for multiplicative relations to make sense of these calculations. In this search, he found three pairs of multiplicative relations in this situation (24÷24=1 and 14.4÷24=0.6; 1x24=24 and 0.6x24=14.4; 100x0.6=60 and 24x0.6=14.4) and newly validated his calculations using proportional reasoning based on these relations. When taking account of the facts that Takuya first wrote the given numbers in PM and that he calculated 60÷24 and 0.6÷24 as well as 24x0.6=14.4, he might find from this experience that relative positions of numbers in PM could suggest necessary calculations and that multiplicative relations in PM could validate those calculations.

In fact, Takuya followed this strategy in most of his solutions in Lesson 6 and 7. He wrote the given numbers in PMs and chose the necessary calculations from the plausible calculations like 500÷0.05 and 500x0.05. After writing the answers in PMs, he added arrows to make explicit the multiplicative relationships in PM and validate the chosen calculations and the answers. When he knew in the class discussions other arrows than what he had written, Takuya also added those new arrows to his PMs. His intentional use of proportional reasoning at the end of Lesson 7, which was discussed in the previous section, was observed after the sequence of such Takuya's solutions. This implies that Takuya's use of PM to solve proportion problems was not a mere internalization or imitation of the teachers' use of PM (see Figure 1), but was supported by his coordination of two functions of PMs, i.e. suggesting necessary calculations based on relative positions of numbers and helping to find multiplicative relationships.

### ***Features of Takuya's Use of PM: Consistent Understanding of the Situations***

As the discussion in the previous section shows, Takuya's use of PM was based on relative positions of numbers and/or on multiplicative relationships illustrated on PM. While they supported different strategies for solving problems about proportions and percent, these two usages of PM have a common feature: searching for a consistency among the elements of a problem situation.

When he used PM to comprehend multiplicative relations in a problem situation, Takuya tried to find and examine relations which were not necessarily required for his solution. In finding 60% of 24m$^2$ in Lesson 5 (Figure 10), Takuya searched for multiplicative relations even after he found an answer by calculating 24x0.6=14.4 and checked it by calculating 14.4÷24=0.6. First he added an arrow from 14.4 to 0.6 and an arrow from 24 to 1 to PM in order to record the latter calculation. His further search led him to finding an arrow from 0.6 to 14.4 and an arrow from 1 to 24, which were the reverse of the previous arrows and could validate his calculation 24x0.6 or 0.6x24 directly. Finally, he also found a horizontal arc between 60 and 100, which validated his calculation in another way. These multiplicative relations were consistent with each other and with the given and the derived numbers, and this consistency totally validated his calculation and answer.

In finding the riding capacity of a railroad car at the end of Lesson 6, Takuya calculated 120÷1.2=100 after he had finished the calculation required to find the answer, 102÷1.2=85. Instead of checking his answer by calculating 85x1.2=102, Takuya calculated120÷1.2=100. He examined whether the multiplicative relationship, ÷1.2, which held between 102 and 85 also held between 120% and 100%. In other words, Takuya checked his answer by examining a consistency among the elements. His use of multiple relations was also observed in Lesson 7. He added both vertical and horizontal arrows in solving Problem 4 (Figure 15). In his solution of Problem 2, Takuya used only horizontal arrows (Figure 14), but he interpreted the relation between 0.2 and 1 and that between 160 and 800 in two ways: "x5" or "÷5"; and "x0.2." Such multiple relations and the consistency among them might give him clearer pictures of problem situations and enable him to make sense of problem situations multilaterally.

His multilateral comprehension of multiplicative relations can explain why Takuya performed some calculations which were seemingly not required to solve problems. When finding 24% of what is 48dL in Problem 5 in Lesson 7, Takuya calculated 0.24x5 and 0.24x4.5 as well as 48÷0.24. These calculations can be considered a part of his attempt to find r so that 0.24xr=1. Such a number r could have enabled him to make sense of the relationship between 0.24 and 1 or between 24% and 100% by "x r" in addition to "÷0.24" and to figure out the base quantity by 48xr. In solving Problem 3 in Lesson 7, Takuya calculated 0.45x2 and 0.45x2.5 after he found the answer, 45%. Even after he had found the answer, Takuya desired to find a multiplicative relationship between the proportion he had figured out and the base "1." These calculations implied that Takuya tried to interpret the relationships among numbers in problem situations in various ways.

When he used PM to visualize relative positions of numbers, Takuya chose the calculation and the answer which made consistent the relationships among the numbers in a problem situation. Takuya always paid his attention to the relative positions of the numbers in PM and he often modified the positions of numbers on PM so that their relative positions became more appropriate. For example, in Lesson 4, Takuya modified the position of "63" on PM when someone said that 63 is less than half in the class discussion. When finding the proportion of 12m to 15m in Lesson 10, he modified the position of 12m on PM and changed it into more appropriate one as 0.8 of 15m. Placing numbers at appropriate positions in terms of relative relationships enabled PMs to show the multiplicative relationships in a rather qualitative manner (Behr et al., 1992) and each position in PMs functioned as a kind of benchmark like "to this extent," even though the relative positions did not show those relationships quantitatively using precise numbers like "x0.2."

Takuya's strategy in which he chose the calculations based on the relative positions of numbers is similar to "Compute and Check strategy," which Lembke & Reys (1994) observed in their 9th- and 11th-grade participants' responses. However, Takuya did not decide which operation was appropriate based only on whether its answer was larger/less than the given number. For example, in solving Problem 3 in Lesson 7, Takuya changed the position of 72 on PM a few times and related the numbers by multiplicative relationships. This fact implies that he was careful not only of larger/less relations, but also of the relative positions and multiplicative relationships of the numbers in PM. That is, Takuya's decisions about appropriate calculations were based on their reasonableness in terms of their consistency with other elements in problem situations.

As discussed above, whether he used PM to visualize multiplicative relationships among elements in problem situations in a quantitative manner using arrows and descriptions like "x0.2" or in a qualitative manner using relative positions of numbers, Takuya attended to the consistency among elements in problem situations or consistent pictures of problem situations. As Takuya, 5th grade student, had not yet learnt algebraic expressions and equations, he could not use PMs to apply the Rule of Three like Dole's (2000) 8th grade students. Instead, he tried to make sense of problem situations in terms of multiplicative relationships and control his proportional reasoning based on consistent pictures of problem situations. Vertical/horizontal arrows, arrows opposite to each other, alternative multiplicative expressions (e.g. ÷5 versus x0.2), and harmonized relative positions of numbers supported each other and enabled him to have consistent and integrated understandings of problem situations (cf. Nunokawa, 2005). Inclusion of various kinds of arrows and interpretations of those arrows made it possible for him to relate numbers multilaterally and apply proportional reasoning flexibly. This flexibility is an indication of advance in learning proportional reasoning (Steinthorsdottir & Sriraman, 2009). PM functioned not only as a diagram which supported his solutions of certain types of problems, but also as a means for improving his understanding of problem situations (Diezmann & English, 2001; Nunokawa, 2006) and for controlling his use of proportional reasoning.

### *Factors Influencing the Insufficient Use of PM*

Although, when using PMs, he tended to use them to make sense of problem situations, there were also some cases in which Takuya did not use PMs even when he faced difficulties in his solutions in the later lessons. The fact that Takuya started to draw PM after his fruitless effort in solving the Book Problem at the end of Lesson 10 suggests that he considered PM a helpful tool for solving complicated problems. But another fact that he did not use PMs to overcome the difficulties in solving Tax Problem, Absentees-and-Attendees Problem, and Enrolled-Students Problem in Lesson 10, as discussed above, makes it difficult to conclude that Takuya learned to fully take advantage of PMs as a thinking tool for proportional reasoning through these lessons.

As discussed in "Basic Characteristics of the Diagram" Section, PM was adopted to make the students more conscious of their use of proportional reasoning and was expected to finally become a cognitive tool to help student control their use of proportional reasoning. When Takuya completed his PM and wrote, "As 0.2 is 1x0.2, I calculated 1500x0.2 and got 300," in solving the last problem in Lesson 7 (Figure 16), PM worked as a cognitive tool. The key to this use of PM was to attend to the relationship between 1 and 0.2 and interpret it as "x0.2." But the analysis of his learning processes demonstrated some factors which might prevent him from doing such reasoning smoothly and, consequently from using PMs easily.

One of these factors was Takuya's number sense about decimal fractions. It was rather weak. To find 24% of what is 48dL, Takuya calculated 0.25x5 and 0.24x4.5. Only after that, he calculated 48x0.24 and 48÷0.24 and chose the latter based on the relative positions of the numbers. He first tried to find r so that 0.24xr=1, instead of interpreting the relationship between 24% and 100% as "÷0.24." Of course, he could have found the answer using this r by calculating 48xr. But it is more difficult to find this r by a trial-and-check approach as he did than to interpret the relationship between 0.24 and 1 as "÷0.24." This suggests that he had difficulty in noticing the multiplicative relationship 0.24÷0.24=1. When he attempted to relate 0.45 and 1, Takuya calculated 0.45x4, 0.45x2, and 0.45x2.5 instead of calculating 1x0.45=0.45 or 0.45÷0.45=1. He felt this difficulty especially when decimal numbers needed to be used in divisors or multipliers. When finding 20% of 800 yen, Takuya interpreted the relationship between 1 and 0.2 or between 100% and 20% as "x5" and "÷5" instead of as "x0.2" (Figure 14). In this case, he could find r so that 0.2xr=1. But he could not find r so that 1xr=0.2 or 0.2÷r=1. Although the search for such r's enabled him to experience multilateral relationships in PM, this weakness made it a little difficult for him to realize the association of his choice of calculations with his proportional reasoning on PM. His limited understanding of multiplication, division, and decimal concepts (Lo & Watanabe, 1997) was the root of his difficulty in using PMs smoothly to control his proportional reasoning.

Second factor is concerned with the timing of plotting the base "1" in PM. Satoh (2008) analyzed 6th graders' learning of multiplicative structures using dual number lines and pointed out that their conscious use of proportional reasoning was closely related to whether students became to write the base "1" first in number lines. In fact, relationship between the base "1" and other quantities or other proportions is, as mentioned above, the important clue for identifying multiplicative relationships and doing proportional reasoning. Takuya did not always write the base "1" first even in the later lessons. When finding 5% winning tickets of 80 lottery tickets in Lesson 6, he wrote "1" on PM after implementing the calculation and writing the answer on PM (Figure 11). He first wrote "0.8" instead of "1" and modified it into "1." He did not write "1" of PM at all in solving the next problem (Figure 12). This way of using PMs implied that Takuya did not pay enough attention to the important role the base "1" played in PMs.

Because there were some students in the class who had difficulty in constructing PMs, we sometimes provided the students with the worksheets in which the base "1" of PMs were printed from the outset. For example, the problem which asked to find 30% of what is 90m² in Lesson 6 was the first time for the students to find the base quantity and they were required to invent ways of finding it. The base "1" was printed because "1" was expected to be a clue for inventing those ways (Figure 13). However, it might be possible that such PMs with the base "1" printed deprived the students of opportunities to consciously write and use the base "1" for finding multiplicative relationships.

To sum up: Since he did not consider the base "1" as a kind of anchor of PMs and could not easily make sense of the relationships between "1" and proportions in terms of the operations with decimal fractions (e.g. "÷0.24"), it was a little difficult for Takuya to use PMs fluently. Because of this limitation, he was unable to make full use of PMs for deciding the required calculations based on proportional reasoning.

## CONCLUDING REMARKS

Although the immaturity of his use of PM discussed in the previous seciton was observed, his use of proportional reasoning became more conscious and intentional through the use of PM, as discussed above. Takuya became able to control proportional reasoning besides halving and doubling. His thinking observed in the later lessons showed the features which Dole et al. (1997) identified in the thinking of their 8th-, 9th-, and 10th-grade proficient participants: a flexible mixture of estimation, benchmarking, and number and operation sense, along with a variety of strategies. But Takuya used PM to make sense of multiplicative structures of the problem situations rather than to apply the Rule of Three systematically. As the discussion about the features of Takuya's use of PMs showed, he searched for various multiplicative relationships in the problem situations and coherent pictures of those situations in terms of multiplicative relationships. These pictures provided him with a means for control of his solutions (Mesa, 2004).

In their learning of multiplicative structures including proportions and percents, students encountered problem situations and were required to find certain information in those situations (Figure 19 (a)). Prawat (1996) pointed out that "ideas borrowed from the disciplines have this potential to illuminate or open up aspects of the world" (p. 223). When they explored such situations, the students were expected to use the mathematical idea of proportional reasoning as the "the instrument" for "a process of transaction with the environment" (Prawat, 1996, p. 224) (Figure 19 (b)). Takuya's learning in this research showed that the PMs functioned to some extent as a tool for exploring the problem situations with the idea of proportional reasoning (Figure 19 (c)). That is, the students' learning of multiplicative structures with PMs can be considered a form of dual mediation of cognitive tools: The relationship between a student and a problem situation is mediated by a conceptual tool, proportional reasoning, and this triangle is mediated by a representational or thinking tool, PM. This second mediation made it possible for the student to control his use of proportional reasoning.

PM elaborated with multi-relations among numbers shows various aspects of a problem situation having a multiplicative structure. This implies that the formula of proportion is not an inflexible rule but an expression of very natural proportional reasoning and it is incorporated in a consistent picture of a problem situation. PMs include both relations corresponding to within-measure-space ratios and those corresponding to between-measure-space ratios (Lamon, 2007). If we take a proportion as a kind of quantity representing "to what extent" as discussed in "Basic Characteristics of the Diagram" Section and see the lower lines of PMs to be a space of proportions, horizontal arrows correspond to within-measure-spaces ratios and vertical arrows correspond to between-measure-spaces ratios. Moreover, relative positions of numbers on PM illustrate "to what extent" visually and show a sense of proportion. Thus, Takuya's way of using PMs suggested that PMs can be more helpful for exploring problem situations having multiplicative structures and for understanding the concepts of proportion and percent, not only for solving problems about proportions and percent.

At the same time, his insufficient use of PM also gives us some implications which we should take account of when using PMs in our teaching to develop students' proportional reasoning. First, Smooth use of PMs requires use of multiplications and division with decimals to interpret relationships among numbers. Students' number sense about multiplications and division with decimals, especially about the calculations in the forms of 1xr=r and r÷r=1, should be developed before they need to use PMs. Second, "1," the

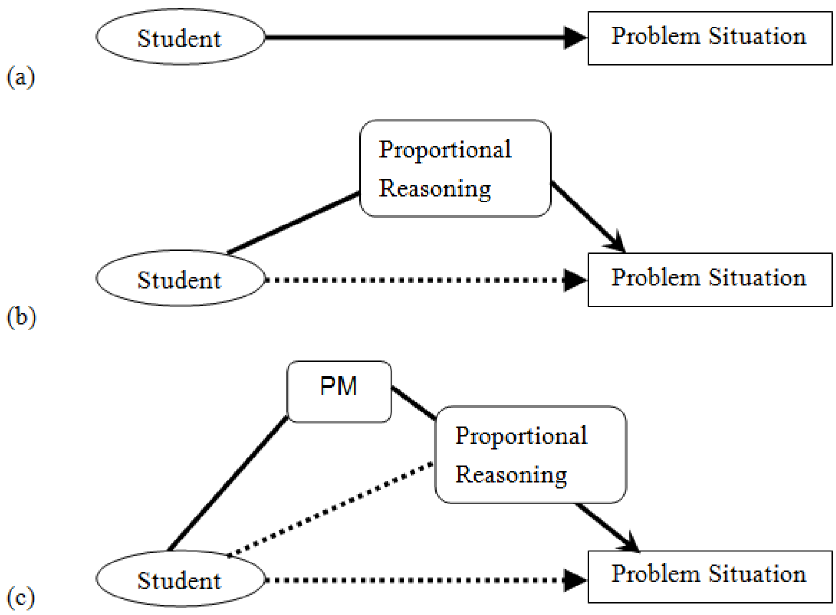



**Figure 19. Double Mediation in Facing Problem Situations**



proportion of a base quantity, plays a critical role when students search for relationships among numbers in PMs. We need to direct students' attention to such critical role of this anchor "1" during their use of PMs. Their attention to "1" also helps students be conscious of the base quantity in finding proportions and understand why a certain quantity called the "base" quantity. Finally, the role of PMs as records of students' reasoning should be also paid attention. If such a representation as PM changes its role from descriptive one to a thinking tool (Gravemeijer, 1997; Van den Heuvel-Panhuizen, 2003), students should be encouraged to represent the multiplicative relationships or proportional reasoning they used even after they found answers. This experience of representing their understanding of problem situations may enable them later to use the same representation as a thinking tool more fluently.

### Note

1) During the class discussion, Takuya erased all he had written on PM here. Figure 4(b) is reproduced from the video data.

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