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# Relational Understanding of the Derivative Concept through Mathematical Modeling: A Case Study

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The purpose of this study was to investigate three second-year graduate students' awareness and understanding of the relationships among the "big ideas" that underlie the concept of derivative through modeling tasks and Skemp's distinction between relational and instrumental understanding. The modeling tasks consisting of warm-up, model-eliciting, and model-exploration activities were used to stimulate participants to reflect and construct ideas about the concept of change. The data indicated that the participants' understanding of derivative was rather instrumental. Their explanations couldn't reveal the role of the big ideas regarding the concept of derivative with respect to what they mean, why and how they are related to the derivative and to each other. The results highlighted the fact that even if one of big ideas is ignored, the concept of derivative may not be fully understood relationally due to the compartmentalization of these big ideas in students' conceptual systems. If this happens to be the case, even though students can solve differentiation tasks/problems correctly, which implies procedural understanding, they may not be actually making sense of what the concept of derivative conceptually means.

*Keywords*: Derivative, mathematical modeling, rate of change, relational understanding.

## INTRODUCTION

The importance of conceptual understanding, along with procedural fluency, has been emphasized as part of students' mathematical proficiency (e.g., see Common Core State Standards Initiative, 2010; Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2000). The major problem preventing students from conceptual understanding has been highlighted as compartmentalized learning (Baroody, Feil, & Johnson, 2007; Berry & Nyman, 2003; Galbraith & Haines, 2000; Hiebert & Lefevre, 1987; Kannemeyer, 2005). As Berry and Nyman (2003) indicated, without opportunities to make connections between concepts and the underlying relations, students could have compartmentalized learning. This causes deficiency in the conceptual understanding of important concepts in mathematics (Mahir, 2009). Students could fill this deficiency with procedural understanding and computation would be regarded as the essential

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***State of the literature***

- Conceptual understanding along with procedural fluency has been emphasized as part of students' mathematical proficiency. Students do not usually understand the fundamental concepts/constructs in (school) mathematics conceptually unless they have opportunities to make connections between concepts and the underlying relations.
- Despite its central role in calculus, the concept of derivative is epistemologically difficult for students. Most students have conceptual difficulties regarding derivative in terms of understanding and giving sense to it.
- Mathematical modeling tasks requiring tackling with the big ideas underlying a concept are seen as a valuable guide in order to observe and understand students' ways of thinking.
- There is a need for studies focusing on students' externalization of their thinking and externalization of their conceptualization steps in problem situations.

***Contribution of this paper to the literature***

- This study argues that the concept of derivative should be introduced and developed in relation to the rate of change, the slope of tangent, and the limit. Because, unless a mathematical concept is understood relationally, students compartmentalize the big ideas related to the concept in their conceptual systems and cannot relate them with each other.
- The findings of this study provide evidence that if even one of the big ideas (i.e., the rate of change, the slope of tangent, and the limit) is ignored, the concept of derivative may not be fully understood relationally. From this point of view, some suggestions were given on teaching the derivative such as the contexts the concept of derivative should be considered and represented, how it should be introduced in the textbooks etc.
- The results indicated that certain issues raised in this manuscript regarding the teaching and learning of derivative concept (i.e., textbooks, university entrance exams, the exams used/prepared by teachers, role of teachers) require attention, and that further studies should be conducted for more extensive suggestions.

outcome with little conceptual understanding (Aspinwell & Miller, 1997). An increasing number of studies in mathematics education have begun focusing on students' understanding of fundamental concepts/constructs in (school) mathematics, such as functions and graphs (Ainsworth, 1999), derivative (Haciomeroglu, Aspinwall, & Presmeg, 2010) and limit (Szydlik, 2000). Despite its central role in calculus, the concept of derivative is epistemologically difficult for students (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Furinghetti & Paola, 1991). Most students have conceptual difficulties regarding derivative in terms of understanding and giving sense to it (Bezuidenhout, 1998; Hauger, 2000). For example, even if students can give correctly "the slope of the tangent line at a certain point on a graph" definition of derivative, they make wrong interpretations of this definition (Amit & Vinner, 1990; Ubuz, 2001). In addition, students have problems in conceptualizing and relating the rate of change to the concept of derivative (Bezuidenhout, 1998; Heid, 1988; Orton, 1983). Another conceptual difficulty is on noticing the difference between average rate of change and instantaneous rate of change in relating these concepts to the concept of derivative (Bingölbali, 2008; Orton, 1983). Moreover, students also have difficulty in conceptualizing the role of limit in (*i*) providing an algebraic definition of derivative, (*ii*) understanding how the average rate of change approximates to the instantaneous rate of change and, (*iii*) understanding how the slopes of the secant lines approximate to the slope of the tangent line (Hankiöniemi, 2006; Orton, 1983).

A relational understanding of derivative should include awareness of the big ideas underlying the concept of derivative, namely the rate of change, the slope of tangent and the limit, and relations between them. Although students can solve differentiation problems correctly, they cannot explain derivative by relating it to the rate of change, the slope of tangent, and the limit (Bingölbali, 2008). As an important reason behind these kinds of learning and achievement contradictions, many researchers highlighted the role of memorizing procedures without understanding the underlying big ideas (Henningsen & Stein, 1997; Schoenfeld, 1992; Shield, 1998). For most students, derivative comprises of excessive amount of differentiation rules without reasons (i.e., the instrumental understanding) (Bingölbali, 2008; Thompson, 1994). From this point of view, considering derivative *relationally* is essential to have its conceptual understanding. In this respect, there is a need for studies focusing on students' internal conceptual systems in order to see how students understand and relate the concepts (Erbas et al., 2014; Clement, 2000; Lakoff & Núñez, 2000). Modeling activities, including real-life situations, can be used as a valuable guide for this purpose, in that model-eliciting activities help teachers and researchers to capture their students' mathematical understanding abilities and skills that generally cannot be captured while using traditional word problem solving activities (Erbas et al., 2014; Lesh, Hoover, Hole, Kelly, & Post, 2000). In traditional word problem

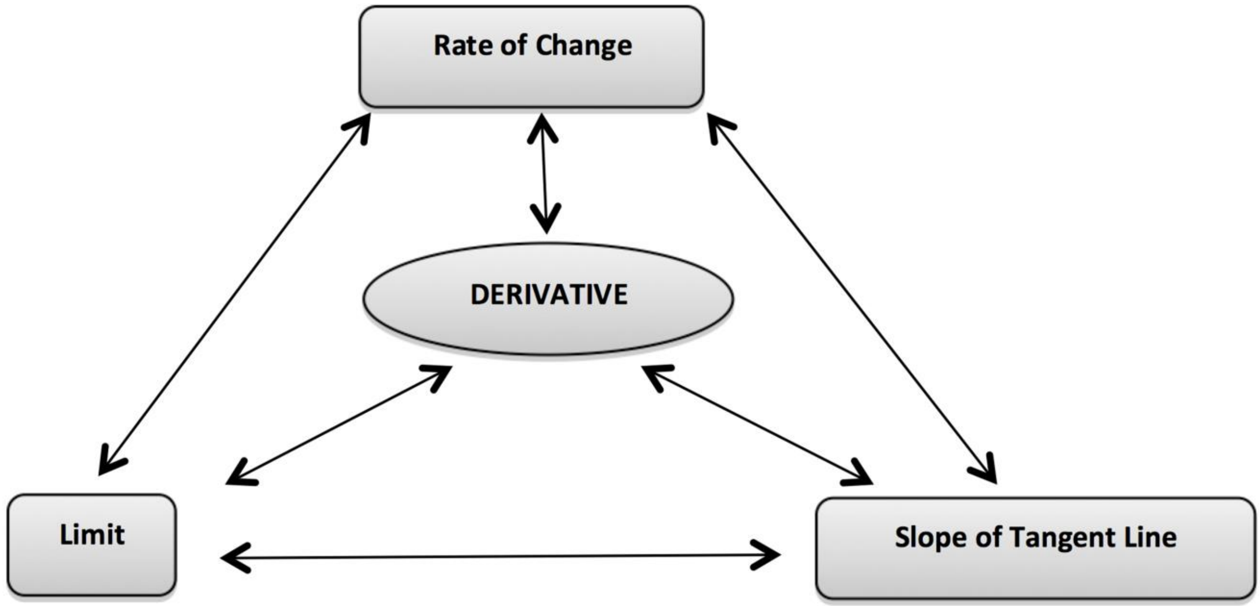


**Figure 1.** Big ideas related to the concept of derivative

solving activities, givens and goals are already specified and words are carefully selected to make apparent the mathematical procedure required for solving the problem (Lesh, Yoon, & Zawojewski, 2006). However, problem-solving activities should be beyond a process where the givens, goals and procedures between them to accomplish an activity are precise (Wyndham & Säljö, 1997). In addition, teachers and researchers are able to understand their students' conceptual strengths and weaknesses during model-eliciting activities to enable them to prepare effective instruction based on these findings (Lesh, Hoover, Hole, Kelly, & Post, 2000). Moreover, model-eliciting activities provide teachers and researchers the opportunity to see students' ways of thinking and possible conceptualization steps (Lesh & English, 2005; Lesh & Sriraman, 2005). Thus, the purpose of this study was to investigate students' understanding of derivative through mathematical modeling tasks that require tackling of the big ideas underlying the concept of derivative and relationships between them.

## THEORETICAL FRAMEWORK

In this study, students' understanding of big ideas regarding the concept of derivative was investigated in the light of Skemp's (1976) theory on conceptualization of mathematical understanding: *relational understanding* (i.e., knowing both what to do and why) and *instrumental understanding* (i.e., knowing rules without reasons). A relational understanding of the concept of derivative demands making sense of certain relations among derivative, the rate of change, the slope of tangent, and the concept of limit (see Figure 1). This requires not only knowing, but also being able to explain the role of these concepts in terms of what it means and why and how it is related to derivative. For instance, a student who understands the concept of derivative can explain how average rate of change approximates to the instantaneous rate of change, as well as how the slopes of secant lines approximate to the slope of tangent line by using the limit concept. Additionally, the student can explain why the instantaneous rate of change at a point is the same as the slope of the tangent at that point. On the other hand, an instrumental understanding of the derivative concept implies knowledge without making sense of what these concepts mean and how they are interrelated in the context of derivative.

## METHODOLOGY

This study utilizes case study design. As described by Creswell (2009), "case studies are a strategy of inquiry in which the researcher explores in depth a program, event, activity, process, or one or more individuals" (p. 13). The case in this study was the phenomenon regarding three graduate mathematics education students' understanding of derivative in terms of their awareness of the big ideas and relationships among them in the context of mathematical modeling.

### Participants

The participants of this study were three second-year mathematics education graduate students in a public university in Ankara, Turkey: Mete (M), Bahar (F), and Meltem (F) (all names are pseudonyms with genders shown in parentheses). As indicated by their CGPAs, the participants were high performers who graduated from mathematics education departments of top ranking universities in Ankara. They were selected purposefully from among eight mathematics education graduate students based on their high computational performance in derivative as measured with tasks requiring the use of derivative concepts and rules (see Appendix B), as well as their mathematical interest and willingness to participate in the study through one-on-one semi-structured interviews conducted prior to the study. The participants expressed that they had studied the concept of derivative during both secondary school and college and received high grades from tests related to the topic. Our purpose for selecting graduate students as participants was that they would be considered as successful and accomplished in the topic at all educational levels and thus investigating the nature of their conceptual systems and the extent to which they would build relational understanding would be more apparent compared to others of less experience and education in the topic. Small groups with three or four students are often recommended in the implementation of mathematical modeling activities in order to develop, describe, explain, manipulate the model, and to control important conceptual systems (Lesh & Yoon, 2004). As a result of this, three participants were decided to be included in the study.

**Modeling Task and Implementation Process**

During the data collection, the participants were asked to work around a "U" shaped table as a group on a mathematical modeling task entitled "An Emergency Patient with High Blood Pressure" (see Appendix A). They were provided with laptop computers with MS Excel spreadsheet software installed, should they wish to use it. The implementation process of the modeling task included three consecutive steps: the warm up activity, the model eliciting activity and the model exploration activity. The first step included participants' reading and discussing a warm-up reading comprised of information from various health-related websites as a way to provide participants with common background information about the context of model eliciting activity (i.e., measuring the blood pressure). This phase took approximately fifteen minutes. The second phase was the model eliciting activity, in which the rate of change concept was integrated into a realistic problem context. In this phase, the participants were asked to read through the problem and individually propose their initial thoughts for the solution approaches and strategies in about ten minutes. After that, the participants were encouraged to focus on the relationships and patterns beyond cursory just the characteristics of the problem in order to generate different ideas and show mathematical usage in real-life situations. This phase intended to uncover participants' ways of thinking while they create models for the problem situation. Including group work, this phase took approximately ninety minutes. In the final phase, that lasted approximately forty-five minutes, the participants were asked to work as a group on a model exploration activity consisting of some evaluation tasks developed to consolidate the fundamental ideas underlying the concept of derivative such as the rate of change, average rate of change and instantaneous rate of change. Researchers assumed the following roles during the implementation of the tasks: During the warm up activity, the researchers tried to check that the participants correctly understood the context of the modeling problem (high blood pressure) by asking probing questions such as "What does high blood pressure mean?" and "When are symptoms of high blood pressure observed?". During the model eliciting and model exploration activities, without directing the participants towards a solution, the researchers attempted to guide them to understand the givens and goals of the problem. In this respect, we asked the participants questions to encourage them to think aloud about how they thought through the modeling task, what kind of ideas they generated, and what solution strategies they proposed (e.g., Why do you think so? How did you come to that conclusion?). In addition, researchers carefully listened to their way of thinking without making any judgments regarding the correctness of the thoughts and comments. However, when they were stuck in their thinking process, researchers provided them with a different point of view by questioning (e.g., How would you think if you would interpret these data graphically rather than numerically in a tabulated form?) and encouraging them to use the spreadsheet software to further their thinking.

**Data Sources and Analysis**

Data sources for the study included field notes (related to participants' explanations that could show their types of understanding of derivative concept as relational or instrumental), worksheets that the participants produced while working on the modeling task, and audio and video recordings of the participants working as a group. For data analysis, firstly, the audio-records were transcribed. Then, to make general sense about participants' understanding of derivative from the data, researchers separately coded and analyzed the transcriptions, worksheets, and field notes in the light of Skemp's (1976) notion of relational understanding of mathematics and the three big ideas related to the concept of derivative. After each researcher highlighted the sections related to the research purposes, these sections were compared and contrasted. Then, consensus was reached based on the discussions on non-agreed sections by each researcher. After inter–rater agreement was provided, the data were organized and analyzed accordingly.

## FINDINGS

### Rate of Change in Relation to the Derivative

During the implementation of the modeling task, it was observed that none of the participants were able to make sense of the rate of change concept. Among the possible reasons for this might be their recollection of prior knowledge and their difficulty in making sense of derivative in a real-life context, other than the well-known velocity-time context that students encounter in most textbooks. Although the context of the model eliciting activity (i.e., the High Blood Pressure) was not related to speed and velocity, participants preferred to use the term "the speed of change" in their attempt to understand the problem. However, they were not only unsure regarding when to use the terms speed and acceleration, but also what these concepts are and how they differ in nature. More precisely, they sometimes interpreted derivative as speed and sometimes as acceleration, as they perceived that the rate of change is the same thing as the speed of change. At the same time, it was observed that the participants could not make sense of the rate of change, unless the term "the speed of change" was used instead. The following excerpt from a dialogue among the participants indicates that they tried to interpret derivative in the speed-time context, but did not explain their interpretations and did not make sense of the term "the rate of change".

Researcher 1: What does the slope of tangent line mean?

Bahar: I think it [*the slope of the tangent line*] is like acceleration in physics...

Meltem: Yes... I think so too.

Mete: [*By trying to interpret based on previous memorized knowledge, while the differentiation of the distance according to the time gives speed, the differentiation of the speed according to the time gives the acceleration*] The first derivative corresponds to the speed, and the second to the acceleration.

Researcher 2: In this problem [*in the context of the model eliciting activity*], what do you mean when you say the acceleration; to what does it correspond?

Mete: I guess the slope of the tangent is the acceleration

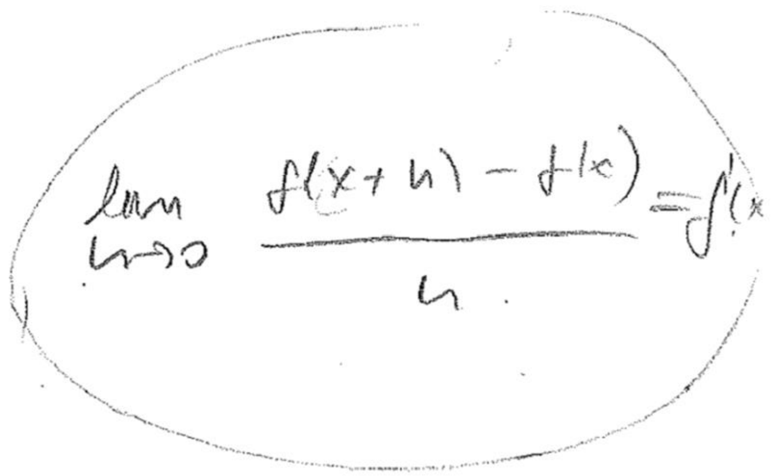

**Figure 2.** Meltem's use of limit in the algebraic definition of derivative

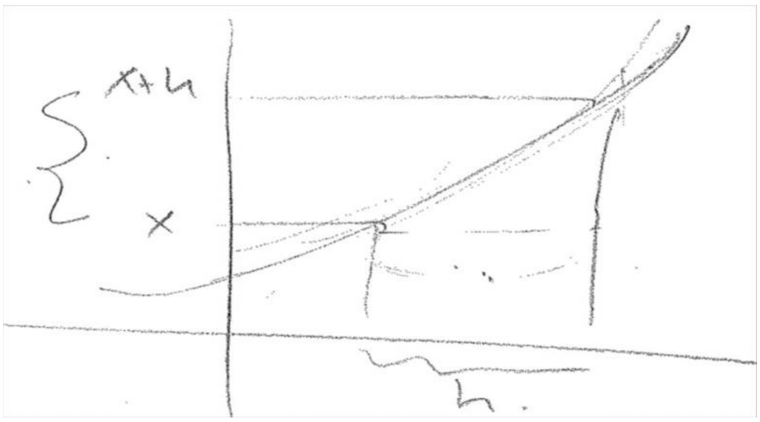


**Figure 3.** Bahar's graphical/geometrical explanation of derivative

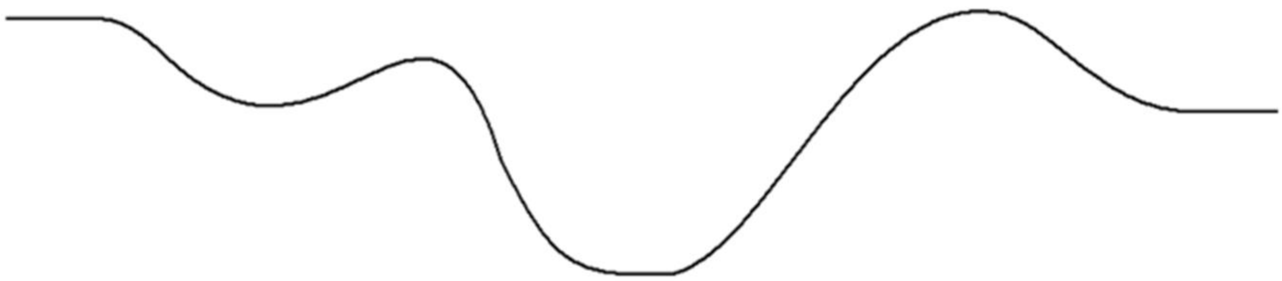

**Figure 4.** Distance-height graph of the Roller Coaster Path in the model exploration activity where students were asked to provide the graph of the first derivative

Bahar: If we investigate the speed of change, it corresponds to the acceleration [*Meltem is confirming this explanation by nodding her head.*]
Researcher 2: What is the speed?
Mete: The distance per unit time
Researcher 1: Where is the distance in this problem?
Mete: ...hmm…
Bahar: There is no distance here, but we only look at the decreasing speed of the blood pressure.
Meltem: [*When reading the term "the rate of change" in the model eliciting activity*] Yeah...this...rate of change...It is related to derivative, isn't it?
Bahar: It's obviously the speed of change...

**Limit in Relation to the Derivative**

In the implementation of the modeling task, participants presented two main difficulties in making sense of the role of the concept of limit in conceptualizing derivative. Their understanding of the limit in the context of derivative was limited to knowing the rule in the algebraic definition of derivative (see Figure 2). Therefore, the approximation of the average rate of change to the instantaneous rate of change had not made sense to them. On the other hand, they couldn't explain the role of the limit in the geometric definition of derivative, in terms of the approximation of the slope of the secant lines to the slope of the tangent line. When trying to explain the geometric definition of derivative, they could only draw a graph based on their previous-knowledge where representations related to the slope such as the secant line and the tangent line were not placed on the graph (see Figure 3). The following excerpt from dialogue among the participants indicates that they had difficulties in making sense of the role of the concept of limit in conceptualizing derivative.

Meltem: You remember... derivative was given with the limit formula… [*when writing the expression in Figure 2*]
Bahar: Right, even there was a graph [*when drawing the graph in Figure 3*] on which derivative was explained.
Mete: Even here [*pointing to the graph drawn by Bahar in Figure 3*], the slope of the tangent line was the derivative...

**Slope of Tangent in Relation to the Derivative**

In the implementation, although the participants defined derivative correctly as "the slope of a tangent line which is drawn to the curve at a certain point", it was observed that they could not make sense of the relation between the slope of the tangent and the derivative at a point, since they interpreted this definition as "equation of the tangent line at a certain point as a derivative of a function". The following excerpt from dialogue among the participants in the implementation of the model eliciting activity indicates that the participants correctly provided the definition of derivative as mentioned above.

Researcher 1: How do you determine the rate of change in blood pressure?
Bahar: With respect to the slope of the tangent line.
Researcher 2: Why?
Meltem: Because the slope of the tangent line was the derivative.
Mete: Yes.

Moreover, the following excerpt from their dialogue, when they were trying to draw the graph of the first derivative of the given function in the model exploration activity (see Figure 4), indicates that the participants' interpretation of this definition was lacking.

Meltem: How is the first derivative function's graph?
Bahar: Hmm... The first derivative function's graph…
Mete: It [*referring to the first derivative function*] is the line graph… …of the slope of the tangent line. Because derivative is the slope of the tangent line, so it should be a line.
Meltem: Right.

**Rate of Change in Relation to the Limit**

Since the participants' conceptualization of the concept of rate of change was "the speed of change",

 

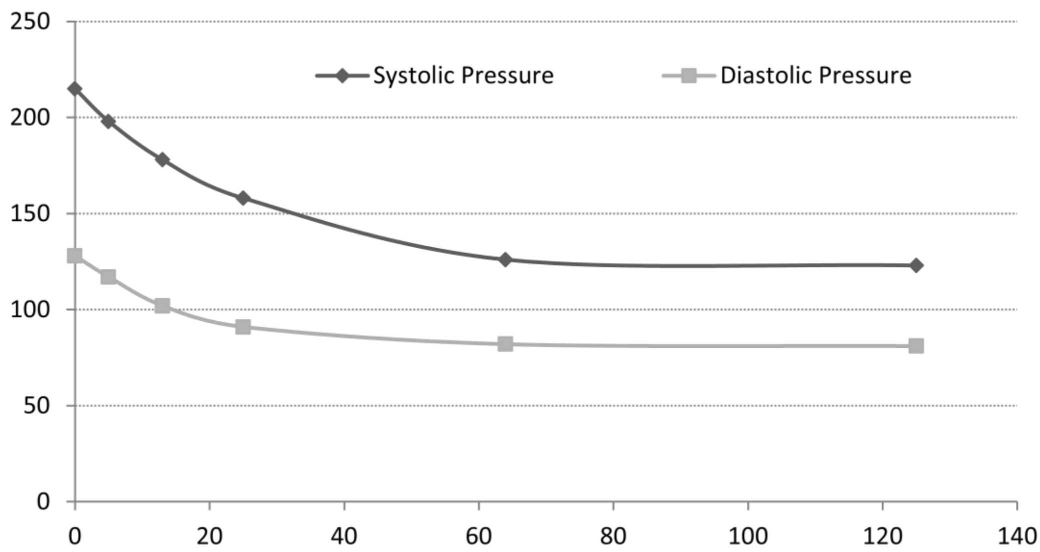


**Figure 5.** Time vs. blood pressure graph produced by the participants in MS Excel

they could not integrate the limit concept into the rate of change. That is to say, they could not realize that with the help of limit, the average rate of change approximates the instantaneous rate of change. The following excerpt from the participants' dialogue while they were trying to explain the algebraic definition of derivative provided in Figure 2, indicated that they did not have a clear interpretation of the concepts of average rate of change and the instantaneous rate of change.

Researcher 2: What does it [*referring to the expression* $\frac{f(x+h)-f(x)}{h}$ *in Figure 2* ] mean?

Meltem: You know... it is like that in the formula...

Researcher 2: Why is this expression in the formula? [*The participants all went quiet for this question for about a minute*.]

Researcher 2: Why does the limit of this expression give derivative? [*All participants again went quiet.*]

**Rate of Change in Relation to the Slope of Tangents**

The participants did not make sense of the concept of the instantaneous rate of change as the slope of the tangent line. Although all three participants defined derivative as the slope of the tangent line at a certain point when interpreting the change in the blood pressure on the graph (see Figure 5), during the process of determining the time of the maximum and minimum rates of decrease in blood pressure, their explanations did not include the concept of instantaneous rate of change since they thought that the change could not be examined instantaneously. The following excerpt shows that the participants lack a conceptual understanding of the concept of instantaneous rate of change.

Bahar: At a certain point, the instantaneous rate of change cannot be mentioned, as the value of blood pressure measurement is already obvious.

Mete: Right, there cannot be a change at a point.

Meltem: I think so.

**Limit in Relation to the Slope of Tangents**

As is mentioned above, when trying to explain the geometric definition of derivative, the participants mentioned none of the representations related to the slope (e.g., the secant line, or the tangent line) on the graph (see Figure 3). Therefore, they could not interpret the role of the limit in the geometric definition of derivative. It was why they did not make sense of why the slope of the secant lines approximate to the slope of the tangent line. The following excerpt from the participants' dialogue while they were trying to explain the geometric definition of derivative on Figure 3 shows this.

Researcher 1: [*Pointing out the two points as A and B in Figure 3*] How would you represent the change on the graph from A to B?

Bahar: … representation of the change on the graph… is it possible?

Mete: I don't know how to show it…

Meltem: Umm... I think change cannot be shown on the graph… [*None of the participants were able to draw the secant line on the graph from point A to point B*]

## DISCUSSION AND CONCLUSIONS

In this study, in spite of the common difficulties of students at all levels in linking word problems to the mathematical ideas (Kaiser, Blomhøj, & Sriraman, 2006; Lesh & Doerr, 2003a, 2003b), none of the participants had difficulty to connect the mathematical concept (i.e., derivative) embedded in the model eliciting and model exploration activities. However, the findings revealed that the participants' understanding of derivative was not relational. First of all, the findings showed that none of the participants realized and explained the meaning of the rate of change and how it is related to the concept of derivative. This result is consistent with those reported in the literature, that both high school and undergraduate students are not making sense of the rate of change and are not aware of the relationship between the derivative and the rate of change concepts (Bezuidenhout, 1998; Bingölbali, 2008; Hauger, 2000; Heid, 1988). As the participants in this study preferred to use the term "speed of change" instead of "rate of change", the semantics or denotation of the word "speed" in Turkish might be considered as an important issue here. The term "speed of change" is used in the meaning of "rate of change" in the previous Turkish national high school mathematics curriculums (Talim ve Terbiye Kurulu Başkanlığı [TTKB], 2005, 2011), before it was subsequently changed in 2013. For the twelfth grade, for example, it was stated that "the derivative is the general name for the instantaneous speed" in an application problem in the context of velocity-time (TTKB, 2011, p. 295). Furthermore, the same terminology is used on some other application problems with different contexts, such as the "speed of change of volume", the "speed of learning" and the "speed of change of unemployment" (TTKB, 2011, p. 302, 312). On the other hand, in the revised current curriculum (TTKB, 2013), the terms "rate of change" and "instantaneous rate of change" are used and students are expected to "describe the rate of change taking advantage of physical and geometrical models" (TTKB, 2013, p. 46). However, there is no particular emphasis on the usage of physical and/or geometrical models, other than those in the velocity-time context. According to Bingölbali (2008), a lack of such emphasis during instruction might restrict the association between derivative and change to the velocity context. Indeed, the results of this study indicate that the context of the model eliciting activity (i.e., High Blood Pressure) caused problems for the participants as they tried to reason with the velocity-time context. From this point of view, while teaching the derivative, it would be more appropriate to start and focus on approaches that provide students with real-life contexts related to the rate of change other than the velocity-time one as it has become a prototypical example.

Another important finding of this study is that the participants were unaware of the connection between the average rate of change and the instantaneous rate of change. In other words, they could not relate the rate of change to the concept of limit. Moreover, it was observed that the participants did not make sense of the instantaneous rate of change as the slope of the tangent line. These results are consistent with those of Orton (1983), arguing that the distinction between the average rate of change and instantaneous rate of change may have little meaning to some students. These results indicated that none of the participants in this study knew and could explain the meaning of the rate of change, why the rate of change is related to derivative, and how the rate of change is related to the limit and the slope of tangent line. Therefore, in the light of the theoretical framework used in this study, it can be concluded that the participants' understanding of the rate of change in relation to the concept of derivative was rather instrumental.

On the other hand, the participants provided the definition of derivative as the slope of a tangent line drawn to the curve at a certain point. However, they interpreted the equation of the tangent line at a certain point as the derivative function of the function, and similar findings are also reported by others (Amit & Vinner, 1990; Ubuz, 2001). According to Amit and Vinner (1990) an underlying cause of such a result could be attributed to the role of memorizing/learning the concept of derivative through ignoring important words in the mathematical sense. Another important cause could be the lack of the graph of the derivative function as a visual object directly referring to the derivative function along with the tangent line, which indirectly refers to it. Most textbooks in both high school level (Ünlü & Er, 2013) and undergraduate level (Steward, 2003), after introducing the symbolic/algebraic definition, provide the geometric interpretation that the slope of the tangent line to the graph of a function *f* at a certain point is defined as the derivative of the function at that point. In this geometric interpretation, however, there is no geometric construction (i.e., the graph of the derivative function) directly referring to the derivative function. Instead, there is only the geometric construction (i.e. the tangent line) referring to the derivative function indirectly. From this point of view, while introducing the geometric definition of the derivative concept in the textbooks, it might be better to emphasize derivative as a function and construct its graph through associating the slopes of different tangent lines of the curve representing the different slopes. In the grand scheme of things, considering that connections among big ideas underlying a concept and its definitions are often hidden in the textbooks (Lithner, 2004; Raman, 2004) and mathematics textbooks are the main source of reference for both students and teachers (Poisson, 2011), it would be vital to provide connections with the big ideas involved with such a concept.

An additional result on the slope of tangent was that the participants did not make sense of the approximation of the slope of the secant lines to the slope of the tangent line. Thus, they could not interpret the role of the limit in the geometric definition of derivative. These results regarding the slope of the tangent line indicated that although the participants knew the definition of derivative as the slope of tangent, none of them could explain the meaning of the slope of the tangent line in terms of why and how it is related to the derivative and to the limit. This implied that the participants' understanding of the slope of the tangent as related to the concept of derivative was rather instrumental.

Finally, the findings of this study regarding the limit revealed that despite participants knowing of the existence of the limit formula in the algebraic definition of derivative, they could not make sense of the role of limit in both the algebraic and geometric definitions of derivative. The reason for this might be the fact that students have difficulties in solving problems that require using the relationship between the derivative and limit, as indicated by Orton (1983). From Skemp's (1976) point of view regarding mathematical understanding, these results indicate that the participants' understanding of the limit concept in relation to the concept of derivative was not relational.

In this study, it can also be concluded that the participants' understanding of derivative was not relational. They could not explain the role of the big ideas related to the concept of derivative, despite the fact that they had been successful in the courses related to derivative. Although there might be other factors, this could be explained by the way derivative is handled/emphasized in exams. In the university entrance exams, for instance, solving mathematical tasks require students often to use reasoning founded on copying algorithms or recalling facts (Bergqvist, 2007; Köğce, 2005). Moreover, the exams used/prepared by teachers to evaluate their students' success at any level depend mostly on procedural understanding instead of conceptual understanding (Flemming & Chambers, 1983; Senk, Beckmann, & Thompson, 1997). Thus, teachers and stakeholders should support the use of exams composed of well-balanced tasks measuring conceptual and procedural knowledge. Moreover, teachers should use problems involving less mathematical terms and put translation of daily life language into mathematical language in order to increase their students' conceptual understanding (Battye & Challis, 1997; English, 2003; English & Lesh, 2003; Gravemeijer & Doorman, 1999; Lesh & Doerr, 2003a, 2003b).

Although this study is limited in terms of the number of participants, the results indicate that issues raised require further attention, and that studies should be conducted on more extensive suggestions. For example, this study provides evidence to some degree that further research support is required in the role of properly selected problem situations (tasks) and proper application strategies that enable teachers to investigate their students' understanding and gain insights into what their students know so that they can attempt to remedy any issues, if they exist, and to prevent compartmentalization of the big ideas in students' conceptual systems for the improvement of students' (relational) understanding. Since, unless a mathematical concept is understood relationally, students may well compartmentalize the big ideas related to the concept in their conceptual systems and therefore cannot relate them with each other. This study claims that the concept of derivative should be introduced and developed in relation to three big ideas (i.e., the rate of change, the slope of the tangent, and the limit). The findings of this study provide evidence that if even one of these big ideas is ignored, the concept of derivative may not be fully understood relationally due to the compartmentalization of these big ideas in students' conceptual systems. If this happens to be the case, even though students can correctly solve differentiation problems, they may not be actually making sense of what the concept of derivative truly means.

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**Appendix A:** An Emergency Patient with High Blood Pressure

Assume that you are a doctor in charge of emergency services. In an evening when you were on duty in the emergency room, a patient with symptoms of high blood pressure arrives at the hospital at 21:15. You measured his blood pressure and seeing that it was high, prescribed an appropriate dose of a high blood pressure-lowering drug. Next, you asked a nurse to measure the patient's blood pressure as frequently as possible and to record the measurement time and the value of systolic and diastolic pressures on the patient's chart, and to inform you if the situation changed. After approximately two hours, you went to check on the patient's situation and the nurse handed you the patient chart (as shown below).

| Time of Measurement | Systolic Pressure (mmHg) | Diastolic Pressure (mmHg) |
|---|---|---|
| 21:15 | 215 | 128 |
| 21:20 | 198 | 117 |
| 21:28 | 178 | 102 |
| 21:40 | 158 | 91 |
| 22:19 | 126 | 82 |
| 23:20 | 123 | 81 |

You report to the Chief of the emergency department with detailed information about the patient's status with regard to the following questions:

- How has the patient's blood pressure changed during their period of stay?
- At what times was the rate of change in the blood pressure of the patient the maximum and minimum?
- Approximately when did the symptoms of high blood pressure disappear?
- Emergency room policy states that a patient cannot be discharged unless his blood pressure is considered normal. Do you think that the patient can be discharged? Explain your answer as when they can be discharged, and why.

**Appendix B:** Taks Used in Selecting the Participants

- Find an equation of the tangent line to the curve at the given point.
  - **a)** $y = 1 + 2x - x^3 \quad (1,2)$
  - **b)** $y = \sqrt{2x+1} \quad (4,3)$
  - **c)** $y = \dfrac{(x-1)}{(x-2)} \quad (3,2)$
- Find the derivatives of the following functions.
  - **a)** $y = \dfrac{3x-2}{\sqrt{2x+1}}$
  - **b)** $y = \ln\left|\dfrac{x^2-4}{2x+5}\right|$
  - **c)** $y = \sin\left(\tan\sqrt{1+x^3}\right)$
- Solve the following problems.
  - **a)** If $A$ is the area of a circle with radius $r$ and the circle expands as time passes, find $\dfrac{dA}{dt}$ in terms of $\dfrac{dr}{dt}$.
  - **b)** Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1m/s, how fast is the area of the spill increasing when the radius is 30 meters?
  - **c)** At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4.00 p.m.?

(*Source*: Steward, J. (2003). *Calculus* (5th Ed.). Belmont, CA: Thomson Brooks/Cole)